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$L^\infty(L^\infty)$-boundedness of $DG(p)$-solutions for nonlinear conservation laws with boundary conditions

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$L^\infty(L^\infty)$-BOUNDEDNESS OF DG(p)-SOLUTIONS FOR NONLINEAR CONSERVATION LAWS WITH BOUNDARY CONDITIONS

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ABSTRACT. We prove the $L^\infty(L^\infty)$-boundedness of a higher-order shock-capturing streamline-diffusion DG-method based on polynomials of degree $p \geq 0$ for general scalar conservation laws. The estimate is given for the case of several space dimensions and for conservation laws with initial and boundary conditions.

1. INTRODUCTION

In this paper we extend the analysis of a shock-capturing streamline-diffusion DG-method for hyperbolic conservation laws in several space dimensions which goes back to [JJS95]. The original DG-method from [JJS95] is based on polynomials of maximal degree $p \geq 0$ (DG(p)-method) and is applied to a pure Cauchy problem. Here we formulate the method for scalar conservation laws with initial and boundary conditions. To describe the further features of the method, we recall the following sufficient conditions for convergence of a sequence of approximate solutions [Sze89b, Remark 1.2]:

(1) uniform boundedness in the $L^\infty(L^\infty)$-norm, i.e. L^∞ in time and L^∞ in space,
(2) weak consistency with all entropy inequalities,
(3) strong consistency with the initial condition.

In the case of an unbounded domain, the condition (1) can be replaced by the

(1*) uniform boundedness in the $L^\infty(L^2)$-norm,

which was done in [JJS95]. To the best of our knowledge, the only attempt to prove the $L^\infty(L^\infty)$-boundedness without using a finer auxiliary triangulation is given in [JSH90] for the case $p = 1$. This proof can be extended for $p > 1$ if the shock-capturing terms are defined on finer triangulations [Sze91]. Thanks to (1*) this is not necessary for the DG(p)-method in [JJS95]. Our result presented here uses the skeleton of the proof from [Sze91], which is based on choosing the test functions $v = I_h^p(U^{q-1})$ with a large even number q, where I_h^p is the Lagrange interpolation operator and U denotes the approximate solution. Within this proof we use a new algebraic argument to verify the coercivity of the shock-capturing term when $v = I_h^p(U^{q-1})$.

Let us recall the key points of the DG(p)-method. First, we have to choose the numerical flux on the element boundaries. In contrast to [JJS95], where a

strictly monotone numerical flux is necessary, we may also use monotone numerical fluxes such as Engquist-Osher fluxes. Second, there are two stabilization mechanisms. The DG(p)-method from the last mentioned reference contains a streamline-diffusion term and a residual-based shock-capturing term. The first term adds an anisotropic artificial viscosity and the second one introduces some isotropic artificial viscosity locally where the solution is nonsmooth.

The paper is organized as follows: In Section 2, we prepare some basic material on scalar hyperbolic conservation laws with initial and boundary conditions. Then, in Section 3 we introduce the DG(p)-method under consideration, and in Section 4 the condition (1*) is verified. After this we present our main Theorem 5.1 which is proved in Sections 5 and 6. Here we give some background material on spatial and algebraic numerical ranges and extend the condition (1) to the case $p > 1$.

The entropy consistency and the consistency with the initial condition will be proved in a forthcoming paper.

2. Hyperbolic conservation laws with Boundary Conditions

Let $Q_T = (0, T) \times \Omega \subset \mathbb{R}^{d+1}, T > 0, d \in \mathbb{N}$, be an open time-space domain with boundary $\Sigma_T = (0, T) \times \Gamma$, $\Gamma = \partial\Omega$ and with outward unit normal n. In this time-space domain a point with position $x = (x_1, x_2, \ldots, x_d)^T$ at time $t = x_0$ has the coordinates $\mathcal{X} = (x_0, x)^T$. Standard notation is used for the space of functions of bounded variations $BV(Q_T)$, Lebesgues spaces $L^q(Q_T)$ and Sobolev spaces $W^{l,q}(Q_T)$, $l \in \mathbb{N}$, $1 \le q \le \infty$.

We consider for $u : Q_T \to \mathbb{R}$ the initial-boundary value problem

$$L(u) = \nabla \cdot \mathcal{F}(u) = 0 \text{ in } Q_T, \tag{2.1}$$

$$u(0, \cdot) = u_0 \text{ on } \Omega, \tag{2.2}$$

with the following boundary condition: For all $k \in \mathbb{R}, r \in \Sigma_T$

$$(\text{sign}(\gamma u(r) - k) - \text{sign}(g_D(r) - k)) (f(\gamma u(r)) - f(k)) \cdot n(r) \ge 0, \tag{2.3}$$

where $\mathcal{F} = (\cdot, f)^T : \mathbb{R} \to \mathbb{R}^{d+1}$, $u_0 : \Omega \to \mathbb{R}$, $g_D : \Sigma_T \to \mathbb{R}$ are given smooth functions and $\gamma : Q_T \to \Sigma_T$ denotes a trace operator. The function sign $: \mathbb{R} \to \mathbb{R}$ is defined by

$$\text{sign}(x) = \begin{cases} x/|x|, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

Due to the hyperbolic nature of (2.1), a boundary condition of the form $u = g_D$ on Σ_T usually over-determines the problem. The generalization of the inflow boundary condition (where $f'(g_D) \cdot n \le 0$) for nonlinear f also leads to a problem that is not well-posed. This difficulty does not occur in $(2.1) - (2.3)$, because the solution u_ϵ of

$$-\epsilon \Delta u_\epsilon + L(u_\epsilon) = 0 \text{ in } Q_T,$$

$$u_\epsilon = g_{\epsilon D} \text{ on } \Sigma_T, \tag{2.4}$$

$$u_\epsilon(0, \cdot) = u_{\epsilon 0} \text{ on } \Omega,$$

converges a.e. to a function $u \in BV(Q_T)$ as $\epsilon \to 0$, which satisfies $(2.1) - (2.3)$ [BlRN79, Theorem 1]. Moreover, we can use this vanishing-viscosity method even in the space $L^\infty(Q_T)$. It is possible to define a well-posed initial-boundary value problem, which admits a unique weak entropy solution $u \in L^\infty(Q_T)$ [MNRR96, Definition 7.2, Theorems 7.28, 8.20].

Let us shortly recall the concept of entropy pairs. We say that $\mathcal{Q} = (\eta, q_1, \ldots, q_d)$ is an entropy pair if $\eta : \mathbb{R} \to \mathbb{R}$ is continuous and convex, the entropy flux $q_j : \mathbb{R} \to \mathbb{R}$ is continuous and $\eta, q_1, \ldots, q_d$ satisfy for all $u \in \mathbb{R}$ the compatibility condition

$$\eta'(u)f'_j(u) = q'_j(u). \tag{2.5}$$

For scalar conservation laws this is trivially fulfilled if the entropy flux is defined as

$$q_j(u) = \int_{g_D}^{u} \eta'(r)f'_j(r)\,dr, \quad 1 \le j \le d. \tag{2.6}$$

3. FORMULATION OF THE DG(p)-METHOD

In this section we introduce the DG(p)-method. To discretize $(2.1) - (2.3)$, let $Q_{n,n+1} = (t_n, t_{n+1}) \times \Omega$, $Q_n = \{t_n\} \times \Omega$ for the sequence of discrete time levels $0 = t_0 < t_1 < \cdots < t_N$, $N \in \mathbb{N}$, be a time-space decomposition of Q_T. The boundary is defined by $\Sigma_{n,n+1} = (t_n, t_{n+1}) \times \Gamma$ and $\Sigma_n = \{t_n\} \times \Gamma$.

Consider an affine decomposition $\mathcal{T}_h^n$ of $Q_{n,n+1}$ belonging to a family of quasi-uniform, admissible decompositions of $Q_{n,n+1}$, cf. [EG04, Definition 1.49, 1.53, 1.140] into simplices or quadrilaterals T and write $\mathcal{T}_h = \bigcup_{n \ge 0} \mathcal{T}_h^n$. Let h_T be the diameter of T and h the maximal diameter of all $T \in T_h^n$.

Set

$$W_h^n = \left\{ w \in L^2(Q_{n,n+1}) : w|_T \in \mathbb{P}_p(T) \quad \forall T \in \mathcal{T}_h^n \right\}, \quad W_h = \prod_{n \ge 0} W_h^n, \tag{3.1}$$

where

$$\mathbb{P}_p(T) = \operatorname*{span}_{\alpha \in \mathbb{N}_0^{d+1},\, |\alpha| \le p} \{\mathcal{X}^\alpha\}, \, \mathcal{X} \in T$$

is the space of polynomials of maximal degree p defined on T.

We are now ready to define the DG(p)-method and introduce by $R_{n,n+1}^i$, R_n^i the set of all interior faces of $Q_{n,n+1}$, Q_n and by $\Lambda_{n,n+1}$, Λ_n the set of all boundary faces. We further set $R_{n,n+1} = R_{n,n+1}^i \cup \Lambda_{n,n+1}$ and $R_n = R_n^i \cup \Lambda_n$. In order to be able to describe discontinuous functions, we denote by τ the common face shared by the elements $T^+ = T$ and T^-. We also define the normal vectors $n_T = n_T^+$ and n_T^- on τ. Then we introduce the notation

$$v^\pm(x) = \lim_{\mu \to +0} v(x - \mu n^\pm), \quad v_\pm^n(x) = v(t_n \pm 0, x_1, \ldots, x_d) \tag{3.2}$$

and

$$\{v\} = \frac{1}{2}(v^+ + v^-), \quad \llbracket vn \rrbracket = v^+ n^+ + v^- n^-, \quad \llbracket v \rrbracket = v^+ - v^-. \tag{3.3}$$

If v is a function on $\Lambda_{n,n+1}$ or Λ_n, we write $v^- = g_D$. By C we will denote a positive constant independent of h, not necessary the same at each occurrence.

Introducing the bivariate form

$$a(v,w) = \sum_{n=0}^{N-1} \sum_{T \in \mathcal{T}_h^n} \left\{ \int_T L(v) w \, d\mathcal{X} + \int_{\partial T} (\hat{\mathcal{F}}(v) - \mathcal{F}(v^+) \cdot n^+) w^+ \, ds \right\}, \tag{3.4}$$

the DG(p)-method for $(2.1) - (2.3)$ can be formulated now: Find $U \in W_h$ such that for $n = 0, 1, \cdots, N$, $U \equiv U_{Q_{n,n+1}} \in W_h^n$ and for all $v \in W_h^n$

$$a(U,v) + \sum_{T \in \mathcal{T}_h^n} \left\{ \left(\delta L(U), \mathcal{F}'(U) \cdot \nabla v \right)_{0,T} + \hat{\epsilon} \left(\nabla U, \nabla v \right)_{0,T} \right\} = 0, \tag{3.5}$$

where

$$\delta = \delta(U) = C_1 h_T \left(\|\mathcal{F}'(U)\|_{l^2} \right)^{-1},$$

$$\hat{\epsilon} = \hat{\epsilon}(U) = \max \left(C_2 h^{2-\beta} R(U), C_3 h^{p+1/2} \right), \quad 0 < \beta < \frac{1}{2},$$

$$R(U)|_T = \max_T \left(|L(U)| \right) + \frac{1}{h_T} \left(\max_{\partial^* T} \left(|[\![\mathcal{F}(U)n]\!]| \right) + \max_{\partial^* T} \left(C_T \, |[\![U]\!]| \right) \right),$$

$$\partial^* T = \{ x \in \partial T : x \notin R_{n+1} \}$$

and $C_1, C_2, C_3 > 0$. As mentioned in the introduction, (3.5) contains a streamline-diffusion term and a residual-based shock-capturing term. Due to the h-dependency of this term, the DG-method can be interpreted as a discrete vanishing-viscosity method.

The numerical flux $\hat{\mathcal{F}}(U)$ is given by

$$\hat{\mathcal{F}}(U) = \{\mathcal{F}(U)\} \cdot n^+ + C_T(U^+, U^-, n^+)[\![U]\!] \tag{3.6}$$

and

$$C_T(v^+, v^-, n^+) = \begin{cases} \frac{1}{2} & n^+ = \pm(1, 0, \ldots, 0), \\ C_0^{\partial\Omega} \geq \int_0^1 |\mathcal{F}'(v^- + s[\![v]\!]) \cdot n^+| \, ds & v^- = g_D, \\ C_0^\Omega \geq \frac{1}{2} \int_0^1 |\mathcal{F}'(v^- + s[\![v]\!]) \cdot n^+| \, ds & \text{otherwise.} \end{cases} \tag{3.7}$$

Within this framework there are the following well-known numerical fluxes: The Engquist-Osher flux if $f(0) = 0$:

$$C_T(v^+, v^-, n^+) = \begin{cases} C_0^{\partial\Omega} = \int_0^1 |\mathcal{F}'(v^- + s[\![v]\!]) \cdot n^+| \, ds & v^- = g_D, \\ C_0^\Omega = \frac{1}{2} C_0^{\partial\Omega} & \text{otherwise.} \end{cases} \tag{3.8}$$

The Lax-Friedrichs flux:

$$C_T(v^+, v^-, n^+) = \begin{cases} C_0^{\partial\Omega} = \sup_{z \in [v^+, v^-]} |\mathcal{F}'(z) \cdot n^+| & v^- = g_D, \\ C_0^\Omega = \frac{1}{2} C_0^{\partial\Omega} & \text{otherwise.} \end{cases} \tag{3.9}$$

Further C_0 is a positive constant satisfying

$$C_0 \geq \|\mathcal{F}'\|_{0,\infty,\mathbb{R}} = \max_{x\in\mathbb{R}}\|\mathcal{F}'(x)\|_{l^2} \tag{3.10}$$

and

$$C_0^{\partial\Omega}, \, C_0^{\Omega} \leq C_0. \tag{3.11}$$

4. PRELIMINARIES

In this section we want to verify the uniform $L^\infty(L^2)$-bound (1*) which is based on [JJS95]. As a byproduct of the study of this technical result, we introduce some notation and prepare the basic material for the $L^\infty(L^\infty)$-bound presented in Section 5. To make this precise, we choose $\eta(U) = U^2/2$, $\varphi = 1$ and $v = \eta'(U)\varphi$. The main theorem of this paper can be obtained by applying the test function $v = I_h^p(\eta'(U)\varphi)$, where $\eta(U) = U^q/q$, $\varphi = 1$ and $I_h^p : C(\overline{Q_T}) \to W_h$ is the Lagrange interpolation operator. Here, due to the fact that $\eta'(U) \notin W_h$, an interpolation or projection operator is necessary. Notice that in this case we get an additional difficulty to estimate terms which contain the difference $\eta'(U)\varphi - I_h^p(\eta'(U)\varphi)$.

By the definition of the bivariate form

$$b(v,w) = a(v,w) + \sum_{n=0}^{N-1}\sum_{T\in\mathcal{T}_h^n}\left(\delta(v)L(v), \mathcal{F}'(v)\cdot\nabla w\right)_{0,T} \tag{4.1}$$

we have, for $w = \eta'(U)\varphi$ and an entropy pair (η, q) satisfying (2.5) and (2.6), that

$$b(v,\eta'(v)\varphi) = \sum_{n=0}^{N-1}\sum_{T\in\mathcal{T}_h^n}\Bigg\{ \left(\delta(v)L(v), \mathcal{F}'(v)\cdot\nabla(\eta'(v)\varphi)\right)_{0,T}$$

$$+ \int_{\partial T}\frac{1}{2}[\![\mathcal{F}(v)n]\!]\eta'(v)\varphi\,ds + \int_{\partial T}C_T[\![v]\!]\eta'(v)\varphi\,ds\Bigg\} \tag{4.2}$$

$$+ \int_{\partial T}\mathcal{Q}(v)\cdot n\varphi\,ds - \int_{[0,t_N]\times\Omega}\mathcal{Q}(v)\cdot\nabla\varphi\,d\mathcal{X}.$$

As usual in DG-methods we consider the different behaviour of inner and boundary faces

$$b(v,\eta'(v)\varphi) = \int_\Omega\eta(v_-^N)\varphi^N\,dx - \int_\Omega\eta(v_-^0)\varphi^0\,dx$$

$$- \int_{[0,t_N]\times\Omega}\mathcal{Q}(v)\cdot\nabla\varphi\,d\mathcal{X} \tag{4.3}$$

$$+ \sum_{i=0}^5 E_i(f,\eta,v,\varphi) - F(f,\eta,v,\varphi),$$

where

$$E_0(f,\eta,v,\varphi) = \sum_{n=0}^{N-1}\sum_{T\in\mathcal{T}_h^n}\left(\delta(v)L(v), \mathcal{F}'(v)\cdot\nabla(\eta'(v)\varphi)\right)_{0,T}, \tag{4.4}$$

$$E_1(f, \eta, v, \varphi) = \sum_{n=0}^{N-1} \int_\Omega \left(\eta(v_-^n) - \eta(v_+^n) - \eta'(v_+^n)(v_-^n - v_+^n) \right) \varphi^n \, dx, \qquad (4.5)$$

$$E_2(f, \eta, v, \varphi) = \sum_{n=0}^{N-1} \sum_{\tau \in R_{n,n+1}^i} \int_\tau \left([\![\mathcal{Q}(v)n]\!] - [\![\mathcal{F}(v)n]\!] \{\eta'(v)\} \right) \varphi \, ds, \qquad (4.6)$$

$$E_3(f, \eta, v, \varphi) = \sum_{n=0}^{N-1} \sum_{\tau \in R_{n,n+1}^i} \int_\tau C_0^\Omega \int_0^1 \eta''(v^- + r[\![v]\!]) dr [\![v]\!]^2 \varphi \, ds, \qquad (4.7)$$

$$E_4(f, \eta, v, \varphi) = \sum_{n=0}^{N-1} \sum_{\tau \in \Lambda_{n,n+1}} \int_\tau \left([\![\mathcal{Q}(v)n]\!] - [\![\mathcal{F}(v)n]\!] \{\eta'(v)\} \right) \varphi \, ds, \qquad (4.8)$$

$$E_5(f, \eta, v, \varphi) = \sum_{n=0}^{N-1} \sum_{\tau \in \Lambda_{n,n+1}} \int_\tau C_0^{\partial\Omega} \int_0^1 \eta''(g_D + r[\![v]\!]) dr [\![v]\!]^2 \varphi \, ds, \qquad (4.9)$$

$$F(f, \eta, v, \varphi) = -\sum_{n=0}^{N-1} \sum_{\tau \in \Lambda_{n,n+1}} \int_\tau \left(\frac{1}{2} [\![\mathcal{F}(v)n]\!] + C_0^{\partial\Omega} [\![v]\!] \right) \eta'(g_D) \varphi \, ds.$$

$$(4.10)$$

Next, we will show the nonnegativity of $\sum_1^5 E_i(f, \eta, v, \varphi)$. By the convexity of η, this is true for E_1. In order to treat E_2, we consider the expression

$$\left(\mathcal{Q}(v^+) - \mathcal{Q}(v^-) - \frac{1}{2} \left(\mathcal{F}(v^+) - \mathcal{F}(v^-) \right) \left(\eta'(v^+) + \eta'(v^-) \right) \right) \cdot n^+$$

$$= \left(\int_{v^-}^{v^+} \left(\mathcal{Q}' - \mathcal{F}'\eta' \right) dr + \int_{v^-}^{v^+} \mathcal{F}' \left(\eta' - \frac{1}{2} \left(\eta'(v^+) + \eta'(v^-) \right) \right) dr \right) \cdot n^+$$

$$\underset{(2.5)}{=} \int_0^1 \mathcal{F}'(v^- + s[\![v]\!]) \cdot n^+ \left(\eta'(v^- + s[\![v]\!]) - \frac{1}{2} \left(\eta'(v^+) + \eta'(v^-) \right) \right) [\![v]\!] \, ds.$$

By the properties of convex functions, it follows that

$$\left(\eta'(v^- + s[\![v]\!]) - \eta'(v^-) \right) [\![v]\!] \geq 0$$

and

$$\left(\eta'(v^- + s[\![v]\!]) - \eta'(v^+) \right) [\![v]\!] = \left(\eta'(v^+ - (1-s)[\![v]\!]) - \eta'(v^+) \right) [\![v]\!] \leq 0.$$

Then we have

$$\left| \left(\eta'(v^- + s[\![v]\!]) - \frac{1}{2} \left(\eta'(v^+) + \eta'(v^-) \right) \right) [\![v]\!] \right|$$

$$= \left| \left(\frac{1}{2} \left(\eta'(v^- + s[\![v]\!]) - \eta'(v^-) \right) + \frac{1}{2} \left(\eta'(v^- + s[\![v]\!]) - \eta'(v^+) \right) \right) [\![v]\!] \right|$$

$$\leq \frac{1}{2} \left| \left(\eta'(v^- + s[\![v]\!]) - \eta'(v^-) \right) [\![v]\!] \right| + \frac{1}{2} \left| \left(\eta'(v^- + s[\![v]\!]) - \eta'(v^+) \right) [\![v]\!] \right|$$

$$= \frac{1}{2}\left(\left(\eta'(v^- + s[\![v]\!]) - \eta'(v^-)\right) - \left(\eta'(v^- + s[\![v]\!]) - \eta'(v^+)\right)\right)[\![v]\!]$$

$$= \frac{1}{2}\left(\eta'(v^+) - \eta'(v^-)\right)[\![v]\!] = \frac{1}{2}\int_0^1 \eta''(v^- + r[\![v]\!])\,dr[\![v]\!]^2,$$

which immediately implies that

$$\left|\left(\mathcal{Q}(v^+) - \mathcal{Q}(v^-) - \frac{1}{2}\left(\mathcal{F}(v^+) - \mathcal{F}(v^-)\right)\left(\eta'(v^+) + \eta'(v^-)\right)\right)\cdot n^+\right|$$

$$\leq \int_0^1 \left|\mathcal{F}'(v^- + s[\![v]\!])\cdot n^+\right|\left|\left(\eta'(v^- + s[\![v]\!]) - \frac{1}{2}\left(\eta'(v^+) + \eta'(v^-)\right)\right)[\![v]\!]\right|\,ds$$

$$\leq \frac{1}{2}\int_0^1 \left|\mathcal{F}'(v^- + s[\![v]\!])\cdot n^+\right|\int_0^1 \eta''(v^- + r[\![v]\!])\,dr[\![v]\!]^2\,ds$$

$$\leq \frac{1}{2}\int_0^1 \left|\mathcal{F}'(v^- + s[\![v]\!])\cdot n^+\right|\,ds\int_0^1 \eta''(v^- + r[\![v]\!])\,dr[\![v]\!]^2.$$

Finally, having in mind the fact that $\eta'' \geq 0$, we can use the definitions (4.7) and (3.7)

$$E_2(f,\eta,v,\varphi) + E_3(f,\eta,v,\varphi)$$

$$\geq \sum_{\substack{n=0 \\ \tau \in R^i_{n,n+1}}}^{N-1} \int_\tau \left(C_0^\Omega - \frac{1}{2}\int_0^1 \left|\mathcal{F}'(v^- + \xi[\![v]\!])\cdot n^+\right|d\xi\right)\left(\int_0^1 \eta''(v^- + r[\![v]\!])dr\right)[\![v]\!]^2 \varphi\, ds$$

$$\geq 0,$$

$$(4.11)$$

where we have used nonnegative test functions φ. The same arguments as before lead to $E_4(f,\eta,v,\varphi) + E_5(f,\eta,v,\varphi) \geq 0$.

Remark 4.1. The local condition (3.7)

$$C_0^\Omega - \frac{1}{2}\int_0^1 \left|\mathcal{F}'(v^- + \xi[\![v]\!])\cdot n^+\right|\,d\xi \geq 0$$

for (4.11) allows a smaller constant C_0^Ω than [JJS95, Remark 2.5]

$$C_0^\Omega - \frac{1}{2}\|\mathcal{F}'\|_{0,\infty,\mathbb{R}}\frac{\int_0^1 \int_0^s \left(n''(v^- + r[\![v]\!]) + \eta''(v^+ - r[\![v]\!])\right)\,dr\,ds}{\int_0^1 \eta''(v^- + r[\![v]\!])\,dr} \geq 0.$$

Thus, the corresponding numerical flux $\hat{\mathcal{F}}(U)$ is a monotone flux function ($U^+ \mapsto \hat{\mathcal{F}}(U)$ is increasing and $U^- \mapsto \hat{\mathcal{F}}(U)$ is decreasing). As mentioned in [JJS95], the requirement that $\hat{\mathcal{F}}(U)$ is a strictly monotone numerical flux, e.g.

$$C_0^\Omega - \frac{1}{2}\int_0^1 \left|\mathcal{F}'(v^- + \xi[\![v]\!])\cdot n^+\right|\,d\xi \geq \epsilon > 0,$$

is necessary for (2). More precisely, the condition

$$\sum_{n=0}^{N-1} \sum_{T \in \mathcal{T}_h^n} \int_T hR(U)^2 \, d\mathcal{X} \leq C\|u_0\|_{0,2,\Omega}^2$$

has to be fulfilled.

Let us now consider the equation (3.5) with $v = \eta'(U)\varphi$:

$$b(U, \eta'(U)\varphi) + \sum_{n=0}^{N-1} \sum_{T \in \mathcal{T}_h^n} \hat{\epsilon}(U) \left(\nabla U, \nabla(\eta'(U)\varphi)\right)_{0,T}$$

$$= \int_\Omega \eta(U_-^N)\varphi^N \, dx - \int_\Omega \eta(U_-^0)\varphi^0 \, dx + \sum_{n=0}^{N-1} \sum_{T \in \mathcal{T}_h^n} \hat{\epsilon}(U) \left(\nabla U, \nabla(\eta'(U)\varphi)\right)_{0,T}$$

$$- \int_{[0,t_N] \times \Omega} \mathcal{Q}(U) \cdot \nabla\varphi \, d\mathcal{X} + \sum_{i=0}^{5} E_i(f, \eta, U, \varphi) - F(f, \eta, U, \varphi) = 0. \tag{4.12}$$

Therefore, in the case $\eta(U) = U^2/2$ and $\varphi = 1$, we obtain

$$\frac{1}{2} \int_\Omega (U_-^N)^2 \, dx + \sum_{n=0}^{N-1} \sum_{T \in \mathcal{T}_h^n} \hat{\epsilon}(U) \left(\nabla U, \nabla U\right)_{0,T}$$

$$+ \sum_{i=0}^{5} E_i(f, U^2/2, U, 1) = \frac{1}{2} \int_\Omega (u_0)^2 \, dx + F(f, U^2/2, U, 1), \tag{4.13}$$

where we have used $U_-^0 = u_0$. By the help of Young's inequality we see that

$$F(f, U^2/2, U, 1) \leq \sum_{n=0}^{N-1} \sum_{\tau \in \Lambda_{n,n+1}} \int_\tau \frac{1}{2} C_0^{\partial\Omega} [\![U]\!]^2 + \frac{9}{8} C_0^{\partial\Omega} g_D^2 \, ds$$

$$= F_1(f, U^2/2, U, 1) + F_2(f, g_D^2/2, g_D, 1), \tag{4.14}$$

thus we arrive at $E_4(f, U^2/2, U, 1) + E_5(f, U^2/2, U, 1) - F_1(f, U^2/2, U, 1) \geq 0$ and

$$\frac{1}{2}\|U_-^N\|_{0,2,\Omega}^2 + \sum_{n=0}^{N-1} \sum_{T \in \mathcal{T}_h^n} \|\delta(U)^{1/2} L(U)\|_{0,2,T}^2 \leq \frac{1}{2}\|u_0\|_{0,2,\Omega}^2 + \frac{9}{8} C_0 \|g_D\|_{0,2,\Sigma_T}^2. \tag{4.15}$$

Finally, using for $t_{N-1} \leq t \leq t_N$ and $q \in \mathbb{N}$ the identity

$$\|U(t, \cdot)\|_{0,q,\Omega}^q = \|U_-^N\|_{0,q,\Omega}^q - q \int_t^{t_N} \int_\Omega U^{q-1}(t', x) \, \text{div} \, \mathcal{F}(U(t', x)) \, dxdt', \tag{4.16}$$

Young's inequality yields

$$\|U(t,\cdot)\|_{0,2,\Omega}^2 \le \|U_-^N\|_{0,2,\Omega}^2 + \frac{C_0}{2C_1 \min\limits_{T \in \mathcal{T}_h^{N-1}} \{h_T\}} \int_t^{t_N} \|U(t',\cdot)\|_{0,2,\Omega}^2 \, dt'$$
$$+ 2 \sum_{T \in \mathcal{T}_h^{N-1}} \|\delta(U)^{1/2} L(U)\|_{0,2,T}^2 \tag{4.17}$$

and a Gronwall argument estimates the right-hand side of (4.17) by means of the left-hand side of (4.15). The quasi-uniformity of $\left\{\mathcal{T}_h^{N-1}\right\}_{h>0}$ ensures the boundedness of the Gronwall constant and the following theorem results.

Theorem 4.2. *Let Ω be a domain with a Lipschitz boundary and $\{\mathcal{T}_h\}_{h>0}$ be a quasi-uniform family of decompositions of $(0,T) \times \Omega$. Let U be a solution of (3.5) satisfying the assumptions (3.10) and (3.11). Then there exists a constant $C > 0$ independent of h, such that, for all $t \in (t_{N-1}, t_N)$,*

$$\|U(t,\cdot)\|_{0,2,\Omega} \le C \left(\|u_0\|_{0,2,\Omega} + \|g_D\|_{0,2,\Sigma_T}\right). \tag{4.18}$$

5. $L^\infty(L^\infty)$-BOUNDEDNESS OF DG(p)-SOLUTIONS

As announced above, in this section we prove that $\|U\|_{0,\infty,Q_T}$ is uniformly bounded. The main idea is to control the interpolation error $U^{q-1} - I_h^p(U^{q-1})$ in the second argument of the bivariate form by the aid of the special shock-capturing term. At the end of this section we formulate two corollaries which are consequences of the limiting process $h \to 0$ and of the special case $p = 0$, respectively. Our main result is the following.

Theorem 5.1. *Let Ω be a domain with a Lipschitz boundary and $\{\mathcal{T}_h\}_{h>0}$ be a quasi-uniform family of decompositions of $(0,T) \times \Omega$. Let U be a solution of (3.5) satisfying the assumptions (3.10) and (3.11). Then there exists a constant $C > 0$ independent of h such that*

$$\|U\|_{0,\infty,Q_T} \le C \left(\|u_0\|_{0,\infty,\Omega} + \|g_D\|_{0,\infty,\Sigma_T} + 1\right). \tag{5.1}$$

The proof is based on the next lemma which contains the extension of [Sze91, Lemma 3.3] and [Sze89a, Lemma 4.2] for $p > 1$ and which is proved in the last section.

Lemma 5.2. *For Lagrange finite elements with a shape regular family of meshes $\{\mathcal{T}_h^n\}_{h>0}$ there is a constant $C > 0$ independent of q and h such that for all $v \in W_h$ and $q = 2m$, $m \in \mathbb{N}$:*

$$\left(\nabla v, \nabla I_h^p(v^{q-1})\right)_{0,T} \ge C \int_T \|\nabla v\|_{l^2}^2 \|v\|_{0,\infty,T}^{q-2} \, d\mathcal{X}, \quad \forall T \in \mathcal{T}_h^n. \tag{5.2}$$

Proof of Theorem 5.1. Setting $v = I_h^p(\eta'(U)\varphi)$ in (3.5) with $\eta(v) = v^q/q$, $\varphi = 1$ and $q > 2$ an even natural number, we obtain

$$\frac{1}{q}\int_\Omega \left(U_-^N\right)^q dx + \sum_{i=0}^{5} E_i(f, U^q/q, U, 1) - \left(b(U, U^{q-1}) - b(U, I_h^p(U^{q-1}))\right)$$

$$+ \sum_{n=0}^{N-1} \sum_{T\in\mathcal{T}_h^n} \hat{\epsilon}(U)\left(\nabla U, \nabla I_h^p(U^{q-1})\right)_{0,T} = \frac{1}{q}\int_\Omega (u_0)^q dx + F(f, U^q/q, U, 1).$$

$$(5.3)$$

Here the key point is that the interpolation error is bounded by the isotropic shock-capturing term. To see this, a careful consideration of the interpolation operator I_h^p is necessary.

By

$$\int_{R_{n+1}} \frac{1}{2}[\![\mathcal{F}(U)n]\!] + C_T[\![U]\!]\, dx = 0,$$

we express the interpolation error as

$$\left(b(U, U^{q-1}) - b(U, I_h^p(U^{q-1}))\right)$$

$$\underset{(4.1)}{=} \sum_{n=0}^{N-1} \sum_{T\in\mathcal{T}_h^n} \left\{ \int_T \nabla\cdot\mathcal{F}(U)\left(U^{q-1} - I_h^p(U^{q-1})\right) d\mathcal{X}\right.$$

$$+ \int_T \delta(U)L(U)\mathcal{F}'(U)\cdot\left(\nabla U^{q-1} - \nabla I_h^p\left(U^{q-1}\right)\right) d\mathcal{X}$$

$$\quad(5.4)$$

$$+ \int_{\partial^* T} \frac{1}{2}[\![\mathcal{F}(U)n]\!]\left(U^{q-1} - I_h^p(U^{q-1})\right) ds$$

$$\left. + \int_{\partial^* T} C_T[\![U]\!]\left(U^{q-1} - I_h^p(U^{q-1})\right) ds\right\} = \sum_{n=0}^{N-1} \sum_{T\in\mathcal{T}_h} \sum_{i=1}^{4} A_T^i.$$

Since $U|_T \in \mathbb{P}_p(T)$, we deduce that $|U|_{p+1,\infty,T} = 0$. Arguing as in [Sze91][p. 765], we may write for $q \geq 3$

$$|U^{q-1}|_{p+1,\infty,T} \leq Cq^{p+1}h^{-p+1}\|\nabla U\|_{0,\infty,T}^2\|U\|_{0,\infty,T}^{q-3}.\qquad(5.5)$$

Together with a standard interpolation error, we thus conclude from the first term of (5.4) that

$$\begin{aligned}
\left|A_T^1\right| &\leq \|(I - I_h^p)U^{q-1}\|_{0,\infty,T}\int_T |\nabla\cdot\mathcal{F}(U)|\, d\mathcal{X}\\[4pt]
&\leq Ch_T^{p+1}|U^{q-1}|_{p+1,\infty,T}\int_T |\nabla\cdot\mathcal{F}(U)|\, d\mathcal{X}\\[4pt]
&\underset{(5.5)}{\leq} Cq^{p+1}h_T^2\|\nabla U\|_{0,\infty,T}^2\|U\|_{0,\infty,T}^{q-3}\int_T |\nabla\cdot\mathcal{F}(U)|\, d\mathcal{X}\\[4pt]
&\leq Cq^{p+1}h_T^2\int_{T\cap\{|U|>1\}} \|\nabla U\|_{0,\infty,T}^2\|U\|_{0,\infty,T}^{q-2}|\nabla\cdot\mathcal{F}(U)|\, d\mathcal{X}
\end{aligned}$$

$$+ Cq^{p+1}h_T^2 \int_{T\cap\{|U|\leq 1\}} \|\nabla U\|_{0,\infty,T}^2 |\nabla \cdot \mathcal{F}(U)| \, d\mathcal{X}$$

$$\leq Cq^{p+1}h_T^2 \max_T(|\nabla \cdot \mathcal{F}(U)|)\|U\|_{0,\infty,T}^{q-2} \int_T \|\nabla U\|_{0,\infty,T}^2 \, d\mathcal{X}$$

$$+ Cq^{p+1}h_T^2 \max_T(|\nabla \cdot \mathcal{F}(U)|) \int_T \|\nabla U\|_{0,\infty,T}^2 \, d\mathcal{X}.$$

By the quasi-uniformity of $\{\mathcal{T}_h^n\}_{h>0}$ and an inverse inequality, we obtain that

$$\int_T \|\nabla v\|_{0,\infty,T}^2 \, d\mathcal{X} \leq C \int_T \|\nabla v\|_{l^2}^2 \, d\mathcal{X}. \tag{5.6}$$

Thus we have, by Lemma 5.2,

$$|A_T^1| \leq Cq^{p+1}h_T^2 \max_T(|\nabla \cdot \mathcal{F}(U)|) \left\{ (\nabla U, \nabla I_h^p(U^{q-1}))_{0,T} + \|\nabla U\|_{0,2,T}^2 \right\}.$$

In a similar fashion, we can estimate the complete right-hand side of (5.4). Consequently, by (4.15) we conclude that

$$\left| \left(b(U, U^{q-1}) - b(U, I_h^p(U^{q-1})) \right) \right|$$

$$\leq Cq^{p+1} \sum_{n=0}^{N-1} \sum_{T\in\mathcal{T}_h} h_T^2 R(U) \left(\nabla U, \nabla I_h^p(U^{q-1}) \right)_{0,T} + Ch_T^\beta q^{p+1}. \tag{5.7}$$

Inserting this into (5.3) we obtain

$$\int_\Omega \left(U_-^N\right)^q dx - Cq^{p+2} \sum_{n=0}^{N-1} \sum_{T\in\mathcal{T}_h} h_T^2 R(U) \left(\nabla U, \nabla I_h^p(U^{q-1}) \right)_{0,T}$$

$$+ q \sum_{n=0}^{N-1} \sum_{T\in\mathcal{T}_h^n} \hat{\epsilon}(U) \left(\nabla U, \nabla I_h^p(U^{q-1}) \right)_{0,T} + \left(\delta(U)L(U), \mathcal{F}'(U)\nabla(U^{q-1}) \right)_{0,T}$$

$$+ \sum_{i=4}^5 E_i(f, U^q/q, U, 1) \leq \int_\Omega (u_0)^q \, dx + qF(f, U^q/q, U, 1) + Ch_T^\beta q^{p+2}. \tag{5.8}$$

To proceed with the treatment of interpolation error, it is necessary to require that $Cq^{p+2} \leq h^{-\beta}$, where $0 < \beta < 1/2$. The upper limit of β is introduced due to convergence reasons, cf. [JJS95, Lemma 3.2]. However, this restriction on q does not prevent us to finish this proof by letting $q \to \infty$.

Moreover, we have

$$\int_\Omega \left(U_-^N\right)^q dx + q \sum_{n=0}^{N-1} \sum_{T\in\mathcal{T}_h^n} \left(\delta(U)L(U), \mathcal{F}'(U)\nabla(U^{q-1}) \right)_{0,T}$$

$$+ \sum_{i=4}^5 E_i(f, U^q/q, U, 1) \leq \int_\Omega (u_0)^q \, dx + qF(f, U^q/q, U, 1) + C. \tag{5.9}$$

Note that

$$
E_4(f, U^q/q, U, 1) + E_5(f, U^q/q, U, 1)
$$
$$
\geq \sum_{n=0}^{N-1} \sum_{\tau \in \Lambda_{n,n+1}} \int_\tau \left(C_0^{\partial\Omega} - \frac{1}{2} \int_0^1 \left| \mathcal{F}'(g_D + \xi[\![U]\!]) \cdot n^+ \right| d\xi \right) [\![U^{q-1}]\!][\![U]\!] \, ds,
$$

and thus, by Young's inequality,

$$
F(f, U^q/q, U, 1)
$$
$$
\leq \sum_{n=0}^{N-1} \sum_{\tau \in \Lambda_{n,n+1}} \int_\tau \left(\frac{1}{2} \int_0^1 \left| \mathcal{F}'(g_D + \xi[\![U]\!]) \cdot n^+ \right| d\xi + C_0^{\partial\Omega} \right) |[\![U]\!] g_D^{q-1}| \, ds
$$
$$
\underset{(3.10)}{\leq} \sum_{n=0}^{N-1} \sum_{\tau \in \Lambda_{n,n+1}} \int_\tau \frac{1}{2} C_0^{\partial\Omega} [\![U^{q-1}]\!][\![U]\!] \, ds + \frac{1}{2} C_0^{\partial\Omega}(q-1) q^{-\frac{q}{q-1}} 3^{\frac{q}{q-1}} 2^{\frac{q-2}{q-1}} g_D^q \, ds
$$
$$
= F_1(f, U^q/q, U, 1) + F_2(f, g_D^q/q, g_D, 1). \tag{5.10}
$$

So we obtain

$$
E_4(f, U^q/q, U, 1) + E_5(f, U^q/q, U, 1) - F_1(f, U^q/q, U, 1) \geq 0.
$$

Altogether we get that

$$
\int_\Omega \left(U_-^N \right)^q dx + q \sum_{n=0}^{N-1} \sum_{T \in \mathcal{T}_h^n} \left(\delta(U) L(U), \mathcal{F}'(U) \nabla(U^{q-1}) \right)_{0,T}
$$
$$
\leq \int_\Omega (u_0)^q \, dx + C_0(q-1)(2q)^{-\frac{1}{q-1}} 3^{\frac{q}{q-1}} \|g_D\|_{0,q,\Sigma_T}^q + C. \tag{5.11}
$$

By repeating the arguments given at the end of the previous section, we summarize that

$$
\sup_{t \geq 0} \|U(t,\cdot)\|_{0,q,\Omega} \leq C^{\frac{1}{q}} \left(\|u_0\|_{0,q,\Omega} + 3^{\frac{1}{q-1}} (C_0 q)^{\frac{1}{q}} \|g_D\|_{0,q,\Sigma_T} + C^{\frac{1}{q}} \right) \tag{5.12}
$$

for $4 \leq q \leq Ch^{-\frac{\beta}{p+2}}$. Finally, using an inverse inequality we have that

$$
\|U\|_{0,\infty,Q_T} \leq \left(Cqh^{-1} \right)^{\frac{d}{q}} \sup_{t \geq 0} \|U(t,\cdot)\|_{0,q,\Omega} \leq \left(Ch^{-1-\frac{\beta}{p+2}} \right)^{\frac{d}{q}} \sup_{t \geq 0} \|U(t,\cdot)\|_{0,q,\Omega}
$$
$$
= C^{\frac{d}{q}} \exp \left(Cdq^{-1} \ln \left(h^{-1} \right) \right) \sup_{t \geq 0} \|U(t,\cdot)\|_{0,q,\Omega}.
$$

Setting $q = Ch^{-\frac{\beta}{p+2}}$ we get

$$
\|U\|_{0,\infty,Q_T} \leq C^{dh^{\beta/(p+2)}} \exp \left(Cdh^{\beta/(p+2)} \ln \left(h^{-1} \right) \right) \sup_{t \geq 0} \|U(t,\cdot)\|_{0,q,\Omega}
$$
$$
\leq C \sup_{t \geq 0} \|U(t,\cdot)\|_{0,q,\Omega} \tag{5.13}
$$

in the case of $h \leq 1$, which concludes the proof. $\square$

Corollary 5.3. *Under the assumptions of Theorem 5.1, the estimate*

$$\|U\|_{0,\infty,Q_T} \le \|u_0\|_{0,\infty,\Omega} + \|g_D\|_{0,\infty,\Sigma_T} + 1 \tag{5.14}$$

holds for $h \to 0$.

Proof. Using (5.12) and (5.13) with $h = Cq^{-(p+2)/\beta} \to 0$, the statement immediately follows.   □

Remark 5.4. The $L^\infty(L^\infty)$-boundedness of DG(p)-solutions with $p > 1$ was also considered in [Sze91]. However, since an inequality of the form (5.2) was proved only for the case $p = 1$, the shock-capturing term was realized on finer auxiliary triangulations using polynomials of first degree. Hence, the bound $C^q q^{p+2} \le h^{-\beta}$ is necessary, which is true for $q \le C \ln(1/h)$, cf. [Sze91, (3.16)]. This gives

$$\|U\|_{0,\infty,Q_T} \le C \sup_{t \ge 0} \|U(t,\cdot)\|_{0,q,\Omega},$$

for $h \to 0$, where $C \ne 1$.

Corollary 5.5. *Under the assumptions of Theorem 5.1, the estimate*

$$\|U\|_{0,\infty,Q_T} \le \|u_0\|_{0,\infty,\Omega} + \|g_D\|_{0,\infty,\Sigma_T} \quad \forall h > 0 \tag{5.15}$$

holds for $p = 0$.

Proof. Let $p = 0$. Then we get $\left(b(U, U^{q-1}) - b(U, I_h^p(U^{q-1}))\right) = 0$. Consequently, there is no need for the bound $q \le Ch^{-\frac{\beta}{p+2}}$, and we can conclude with letting $q \to \infty$ in (5.12) and (5.13).   □

6. PROOF OF LEMMA 5.2

Until now there is no proof of an inequality like (5.2) for $q \ne 2$. To the best of our knowledge only special cases for linear ansatz functions on triangles respectively tetrahedrons are available, cf. [Sze89a, Lemma 4.2] and [Sze91, Lemma 3.3]. Moreover, the constant in these references depends on q.

Using the theory of numerical ranges for bounded linear operators in Banach spaces, we are able to prove this inequality under rather weak assumptions. More precisely, the local stiffness matrix of the shock-capturing term has to be symmetric positively definite and an eigenvector $(1, \cdots, 1)^T$ with an unique eigenvalue zero.

First of all we need some further notation and definitions about the numerical range.

Definition 6.1. Let $(X, \|\cdot\|)$ be a normed vector space, let $S(X)$ be the unit sphere and denote by X' the dual space of X. For each bounded linear operator A on X,

$$W(A, \|\cdot\|) = \{f(Ax) : (x,f) \in \Pi\} \tag{6.1}$$

with $\Pi = \{(x,f) \in S(X) \times S(X') : f(x) = 1\}$ is called the spatial numerical range.

Remark 6.2.

(1) Note that the following definition is equivalent to (6.1):

$$W(A, \|\cdot\|) = \{f(Ax) : \ f(x) = \|x\|\|f\| = 1\},\qquad(6.2)$$

since

$$\|f\| = \sup_{\|x\|=1} |f(x)| \le \|f\| \underbrace{\|x\|}_{=1} = 1.$$

(2) In contrast to the spectrum $\sigma(A)$, the spatial numerical range $W(A, \|\cdot\|)$ depends on the norm $\|\cdot\|$.

(3) Let $l^{q_1}(n) = (\mathbb{R}^n, \|\cdot\|_{l^{q_1}})$ be the normed vector space. Due to $f(x) = \sum_{i=1}^n x_i f(e_i) = x^T y_f$ and $l^{q_2}(n)$ for $1/q_1 + 1/q_2 = 1$, which is norm-isomorphic to $l^{q_1}(n)'$, the identity $\|f\| = \|y_f\|_{l^{q_2}}$ is valid. Therefore we get

$$W(A, \|\cdot\|_{l^{q_1}}) = \left\{ x^T A y_f : \ x^T y_f = \|x\|_{l^{q_1}} \|y_f\|_{l^{q_2}} = 1 \right\},\qquad(6.3)$$

which is the identity case of Hölder's inequality, cf. [Bau62]. In the case $q_1 = 2$, $W(A, \|\cdot\|_{l^2}) = W(A)$ is the numerical range in a Hilbert space due to Toeplitz [Toe18].

(4) Unlike to $W(A)$, the spatial numerical range is not necessary convex, cf. [NS64, S. 357].

If we interpret the Matrix A as an element of a normed algebra $\mathcal{A}$ with an identity element, we can define a second numerical range. For further details we refer to [BD71, S. 15].

Definition 6.3. Let $\mathcal{A}$ be a normed algebra, $S(\mathcal{A}) = \{x \in \mathcal{A} : \|x\| = 1\}$ the unit sphere and $\mathcal{A}'$ the dual space of $\mathcal{A}$. For $x \in \mathcal{A}$ let

$$D(\mathcal{A}, x) = \left\{ f \in \mathcal{A}' : f(x) = 1 = \|f\| \right\}.\qquad(6.4)$$

We define the algebraic numerical range by

$$V_\mathcal{A}(a, \|\cdot\|) = \cup \left\{ V_\mathcal{A}(a, x, \|\cdot\|) : x \in S(\mathcal{A}) \right\},\qquad(6.5)$$

where

$$V_\mathcal{A}(a, x, \|\cdot\|) = \{f(ax) : f \in D(\mathcal{A}, x)\}.\qquad(6.6)$$

Notice that for the algebraic numerical range it is sufficient to consider only the identity element.

Lemma 6.4.

$$V_\mathcal{A}(a, \|\cdot\|) = V_\mathcal{A}(a, 1, \|\cdot\|), \ a \in \mathcal{A}.$$

Proof. [BD71, Lemma 2.2]. □

Next, we recall two well-known results about numerical ranges and the connection to the spectrum $\sigma(A)$.

Lemma 6.5.

$$\text{conv } W(A, \|\cdot\|) = V_\mathcal{A}(a, \|\cdot\|).\qquad(6.7)$$

Proof. [BD71, S. 84]. □

Theorem 6.6 (Vidav). *Let $a \in \mathcal{A}$ be a Hermitian element, i.e., $V_\mathcal{A}(a, \|\cdot\|) \subset \mathbb{R}$. Then we have*

$$\operatorname{conv} \sigma(a) = V_\mathcal{A}(a, \|\cdot\|). \tag{6.8}$$

Proof. [BD71, Corollary 5.11]. □

Further we have a corollary which will help us to prove Lemma 5.2.

Corollary 6.7. *Let A be a symmetric, positively semidefinite matrix. Then we have*

$$W(A, \|\cdot\|) \subseteq [0, \lambda_{\max}(A)], \tag{6.9}$$

where $\lambda_{\max}(A)$ denotes the largest eigenvalue of A.

Proof of Lemma 5.2. First, consider (5.2) on the reference element. Obviously, the inequality is valid for $v = \operatorname{const}$. Let $v \neq \operatorname{const}$ be given. Consider a decomposition of $\mathbb{Q}_p(\hat{T}) = V^0(\hat{T}) \oplus V(\hat{T})$ with

$$V^0(\hat{T}) = \{v \in \mathbb{Q}_p(\hat{T}) : v = \operatorname{const}\}$$

$$= \{v \in \mathbb{Q}_p(\hat{T}) : \int_{\hat{T}} \nabla w \cdot \nabla v \, dx = 0, \ w \in \mathbb{Q}_p(\hat{T})\}.$$

Let $V_\mathcal{N}$ and V_N^0, $V_\mathcal{N}^0 \oplus V_\mathcal{N} = \mathbb{R}^{n_{\mathrm{dof}}}$ be the coefficient spaces. Therefore, due to the definition of Lagrange finite elements we have that

$$V_\mathcal{N}^0 = \operatorname{span}\{(1, \ldots, 1)^T\}, \ \dim V_\mathcal{N}^0 = 1. \tag{6.10}$$

Let $\mathcal{N}$ denote the Lagrange nodes and n_{dof} the number of degrees of freedom. Moreover, defining $\nabla \varphi = (\nabla \varphi_1, \ldots, \nabla \varphi_{n_{\mathrm{dof}}})^T$, $v_\mathcal{N} = (v(x))_{x \in \mathcal{N}}$, $v_\mathcal{N}^{q-1} = (v^{q-1}(x))_{x \in \mathcal{N}}$, we have that

$$\frac{\left(\nabla v, \nabla I_h^p(v^{q-1})\right)_{0, \hat{T}}}{\|v_\mathcal{N}\|_{l^q}^q} = \frac{v_\mathcal{N}^T \hat{A} v_\mathcal{N}^{q-1}}{v_\mathcal{N}^T v_\mathcal{N}^{q-1}} = v_\mathcal{N}^T \hat{A} v_\mathcal{N}^{q-1}$$

if $v_\mathcal{N}^T v_\mathcal{N}^{q-1} = 1$. Notice that, due to the homogeneity of the quotient, such a norming is always possible. Using the fact that $\hat{A}$ is a symmetric, positively semidefinite matrix and $1 = v_\mathcal{N}^T v_\mathcal{N}^{q-1} = \|v_\mathcal{N}\|_{l^q} \|v_\mathcal{N}^{q-1}\|_{l^{q/(q-1)}}$, we obtain

$$v_\mathcal{N}^T \hat{A} v_\mathcal{N}^{q-1} \in W(\hat{A}, \|\cdot\|_{l^q}) \underset{(6.9)}{\subseteq} [0, \lambda_{\max}(\hat{A})].$$

Now, it is natural to ask whether $v_\mathcal{N}^T \hat{A} v_\mathcal{N}^{q-1}$ is bounded from zero independent of q. To see this, let us suppose that the eigenvalues are ordered in increasing manner

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{n_{\mathrm{dof}}}.$$

Since $V_\mathcal{N}^0$ is the eigenspace of λ_1, we have

$$V_\mathcal{N}^0 \perp V_\mathcal{N}, \tag{6.11}$$

and we can write $\hat{A}$ in terms of a sum of dyadic products of eigenvectors ξ_i, $1 \leq i \leq n_{\mathrm{dof}}$

$$v_\mathcal{N}^T \hat{A} v_\mathcal{N}^{q-1} = v_\mathcal{N}^T \sum_{i=1}^{n_{\mathrm{dof}}} \lambda_i \xi_i \xi_i^T v_\mathcal{N}^{q-1} = \sum_{i=1}^{n_{\mathrm{dof}}} \lambda_i v_\mathcal{N}^T \xi_i \xi_i^T v_\mathcal{N}^{q-1} = \sum_{i=2}^{n_{\mathrm{dof}}} \lambda_i v_\mathcal{N}^T \xi_i \xi_i^T v_\mathcal{N}^{q-1}.$$

Again, the inclusion (6.9) yields $v_{\mathcal{N}}^T \xi_i \xi_i^T v_{\mathcal{N}}^{q-1} \in [0,1]$. On the other hand, supposing $v_{\mathcal{N}} \neq \mathrm{const}$, we obtain

$$v_{\mathcal{N}}, v_{\mathcal{N}}^{q-1} \perp \mathrm{span}\{\xi_1\} = V_{\mathcal{N}}^0. \tag{6.12}$$

Then, the boundedness of $v_{\mathcal{N}}^T \hat{A} v_{\mathcal{N}}^{q-1}$, i.e.

$$v_{\mathcal{N}}^T \hat{A} v_{\mathcal{N}}^{q-1} \geq \lambda_2 \sum_{i=1}^{n_{\mathrm{dof}}} v_{\mathcal{N}}^T \xi_i \xi_i^T v_{\mathcal{N}}^{q-1} - \lambda_2 \underbrace{v_{\mathcal{N}}^T \xi_1}_{=0} \xi_1^T v_{\mathcal{N}}^{q-1} \geq \lambda_2 v_{\mathcal{N}}^T v_{\mathcal{N}}^{q-1},$$

implies that

$$\frac{\left(\nabla v, \nabla I_h^p(v^{q-1})\right)_{0,\hat{T}}}{\|v_{\mathcal{N}}\|_{l^q}^q} \geq \lambda_2, \quad v \neq \mathrm{const}. \tag{6.13}$$

Now, standard estimates give the proof for the reference element

$$\|v\|_{0,\infty,\hat{T}}^{q-2}\|\nabla v\|_{0,2,\hat{T}}^2 \leq \lambda_{\max}(\hat{A})\|v^{q-2}\|_{0,\infty,\hat{T}}\|v_{\mathcal{N}}\|_{l^2}^2$$

$$\leq \lambda_{\max}(\hat{A})\Lambda_p\|v_{\mathcal{N}}\|_{l^\infty}^{q-2}\|v_{\mathcal{N}}\|_{l^2}^2$$

$$\leq \lambda_{\max}(\hat{A})\Lambda_p\|v_{\mathcal{N}}\|_{l^q}^{q-2}\|v_{\mathcal{N}}\|_{l^2}^2$$

$$\leq (n_{\mathrm{dof}})^{1-2/q}\,\lambda_{\max}(\hat{A})\Lambda_p\|v_{\mathcal{N}}\|_{l^q}^q$$

$$\leq n_{\mathrm{dof}}\frac{\lambda_{\max}(\hat{A})}{\lambda_2}\Lambda_p\left(\nabla v, \nabla I_h^p(v^{q-1})\right)_{0,\hat{T}},$$

where $\Lambda_p = \|\sum_{i=1}^{n_{\mathrm{dof}}}|\varphi_i|\|_{0,\infty,\hat{T}}$ is the Lebesgue constant.

Finally, we want to show the result for an affine decomposition $\{\mathcal{T}_h^n\}_{h>0}$ with

$$F_T : \hat{T} \ni \hat{x} \mapsto J_T\hat{x} + b_T = x \in T \qquad \forall T \in \{T_h^n\}_{h>0}$$

and therefore

$$\nabla u(x) = J_T^{-T}\hat{\nabla}\hat{u}(\hat{x}), \quad \hat{u} = u \circ F_T. \tag{6.14}$$

Due to the spectral decomposition of

$$K = \left(J_T^T J_T\right)^{-1} = \sum_{l=1}^d \mu_l\psi_l\psi_l^T,$$

we have that

$$\int_T \nabla v \cdot \nabla I_h^p(v^{q-1})\,dx = v_{\mathcal{N}}^T\left(\int_T \nabla\varphi_j \cdot \nabla\varphi_i\,dx\right)_{i,j} v_{\mathcal{N}}^{q-1}$$

$$= v_{\mathcal{N}}^T\left(\int_{\hat{T}}(\hat{\nabla}\hat{\varphi}_j)^T K\hat{\nabla}\hat{\varphi}_i|\det(J_T)|\,d\hat{x}\right)_{i,j} v_{\mathcal{N}}^{q-1}$$

$$= |\det(J_T)|\sum_{l=1}^d \mu_i v_{\mathcal{N}}^T\left(\int_{\hat{T}}(\hat{\nabla}\hat{\varphi}_j)^T\psi_l\psi_l^T\hat{\nabla}\hat{\varphi}_i\,d\hat{x}\right)_{i,j} v_{\mathcal{N}}^{q-1}$$

$$\underset{(6.9)}{\geq} |\det(J_T)|\mu_{\min}v_{\mathcal{N}}^T\hat{A}v_{\mathcal{N}}^{q-1}$$

$$= |\det(J_T)|\|J_T\|_{l^2}^{-2}v_{\mathcal{N}}^T\hat{A}v_{\mathcal{N}}^{q-1}$$

and

$$|\hat{\nabla}\hat{v}|^2_{0,2,\hat{T}} \ge |\det(J_T)|^{-1}\|J_T^{-1}\|^{-2}_{l^2}|\nabla v|^2_{0,2,T}.$$

Now, the shape regularity property

$$\|J_T\|_{l^2}\|J_T^{-1}\|_{l^2} \le C$$

completes the proof. □

7. Summary

In this paper, we considered a DG-method based on polynomials of degree $p \ge 0$ for hyperbolic scalar conservation laws. This method was introduced for the pure Cauchy problem in [JJS95]. We extended the formulation for hyperbolic conservation laws with initial and boundary conditions. Moreover, we presented a proof of the uniform boundedness of the discrete solution in the $L^\infty(L^\infty)$-norm. The analysis is based on arguments demonstrated in [Sze91] which are valid for $p = 1$. It turned out that the use of numerical ranges for bounded linear operators in Banach spaces allows to generalize this result to the case $p > 1$. Future work will be devoted to the convergence of the DG-method for the initial-boundary value problem.

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