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Convolution algebras for Heckman-Opdam polynomials derived from compact Grassmannians

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Abstract: We study convolution algebras associated with Heckman-Opdam polynomials. For root systems of type BC we derive three continuous classes of positive convolution algebras (hypergroups) by interpolating the double coset convolution structures of compact Grassmannians U/K with fixed rank over the real, complex or quaternionic numbers. These convolution algebras are linked to explicit positive product formulas for Heckman-Opdam polynomials of type BC, which occur for certain discrete multiplicities as the spherical functions of U/K. These results complement those of a recent paper by the second author for the non-compact case.

Key words: Heckman-Opdam polynomials, Grassmann manifolds, product formula, hypergroup convolution.
AMS subject classification (2000): 33C52, 53C35, 43A62, 33C80.

1 Introduction

In the theory of multivariable hypergeometric functions and polynomials of Heckman, Cherednik and Opdam, the existence of product formulas and positive convolution algebras is in general unsolved. In [18], three continuous series of positive convolution algebras having Heckman-Opdam hypergeometric functions as multiplicative functions were obtained by interpolating geometric cases in an explicit way, namely the product formulas for the spherical functions of non-compact Grassmannians. In these cases, a full picture of harmonic analysis for the hypergeometric transform could thus be obtained. The present paper extends these results to the dual situation related to compact Grassmannians and convolution algebras for three continuous series of Heckman-Opdam Jacobi polynomials of type BC.

To be specific, we consider the compact Grassmann manifolds U/K where $U = SO(p+q)$, $SU(p+q)$ or $Sp(p+q)$ and $K = SO(p)\times SO(q)$, $S(U(p)\times U(q))$ or $Sp(p) \times Sp(q)$, respectively. These are dual to the noncompact Grassmannians studied in [18]. Following the procedure in loc.cit., we write down the product formula for their spherical functions in a way which allows analytic continuation with respect to the dimension parameter p, the rank q being fixed. The spherical functions are Heckman-Opdam Jacobi polynomials of type BC with certain discrete multiplicities, and our continuation gives an explicit product formula for an interpolated continuous range of multiplicities. This formula in part generalizes Koornwinder's product formula for Jacobi polynomials [12] to higher rank. Naturally, it is similar to the non-compact case, but direct analytic continuation from the non-compact to the compact case seems to be not feasible.

The compact case is easier in some respect (identifying the dual space for instance), but also needs some special care when studying the geometric background. We obtain commutative hypergroup algebras on the fundamental alcove of the associated affine reflection group, with the associated Heckman-Opdam Jacobi polynomials as characters.

The organisation of this paper is as follows: In Section 2 we recall some basics of trigonometric Dunkl theory. Section 3 is a short summary of the necessary background from the theory of symmetric spaces. After this we start in Section 4 with the compact Grassmannians U/K, identify their spherical functions with Jacobi polynomials, and use a KAK-type decomposition to make their product formula explicit. Following the idea of [18], this product formula is then analytically continued. Section 5 contains a review of the rank one case, and in Section 6, the related hypergroup structures on the fundamental alcove are studied.

2 Fundamentals of Trigonometric Dunkl Theory

This section is a short review of the fundamentals of trigonometric Dunkl theory which will be needed in this article. For details, we refer to the work of Heckman and Opdam ([6], [15], [16]).

Let $\mathfrak{a}$ be a q-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$, which is extended to a complex bilinear form on the complexification $\mathfrak{a}_{\mathbb{C}}$ of $\mathfrak{a}$. We identify $\mathfrak{a}$ with its dual space $\mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$ via the given inner product. Let $\Sigma \subset \mathfrak{a}$ be a (not necessarily reduced) root system. For $\alpha \in \Sigma$ we write $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ for the coroot of α and denote by $s_\alpha(x) = x - \langle \alpha^\vee, x \rangle \alpha$ the reflection in the hyperplane H_α perpendicular to α.

The reflections $\{s_\alpha : \alpha \in \Sigma\}$ generate the Weyl group $W = W(\Sigma)$. We define the root lattice $Q := \mathbb{Z}.\Sigma$ and the coroot lattice $Q^\vee = \mathbb{Z}.\Sigma^\vee$. Further, we fix some positive subsystem Σ^+ of Σ, as well as a basis $\{\alpha_1, \dots \alpha_q\} \subset \Sigma^+$ of simple roots. An element $\lambda \in \mathfrak{a}$ is called (strictly) dominant, if $\langle \lambda, \alpha_i \rangle \geq 0$ (respectively > 0) for all $i = 1, \dots, q$. We write $\mathfrak{a}^+ := \{\lambda \in \mathfrak{a} : \langle \lambda, \alpha^\vee \rangle >$

$0\ \forall \alpha \in \Sigma^+\}$ for the Weyl chamber of strictly dominant elements.

For $\alpha \in \Sigma$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}$ let

$$\lambda_\alpha := \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

The *weight lattice* is given by

$$\Lambda := \{\lambda \in \mathfrak{a} \,:\, \lambda_\alpha \in \mathbb{Z} \text{ for all } \alpha \in \Sigma\}$$

and the set

$$\Lambda^+ := \{\lambda \in \mathfrak{a} \,:\, \lambda_\alpha \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+\}$$

is called the lattice of dominant weights. Here we use the notation $\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$. The positive root lattice $Q^+ = \mathbb{Z}^+.\Sigma^+$ defines a partial ordering $\preceq$ on $\mathfrak{a}$:

$$\mu \preceq \lambda \iff \lambda - \mu \in Q^+.$$

This ordering is called the dominance ordering.

2.1 Lemma. *Let $\gamma \in \overline{\mathfrak{a}^+}$ be dominant. Then $w\gamma \preceq \gamma$ for all $w \in W$.*

Proof. Lemma 10.3B in [10]. $\square$

2.2 Lemma. *Let $\lambda, \mu \in \Lambda^+$ be dominant weights with $\mu \preceq \lambda$. Then $|\mu| \leq |\lambda|$.*

Proof. Let $\lambda, \mu \in \Lambda^+$ with $\mu \preceq \lambda$. Then $\lambda + \mu$ is also dominant and $\lambda - \mu$ is a sum of positive roots. Therefore

$$0 \leq \langle \lambda + \mu, \lambda - \mu \rangle = |\lambda|^2 - |\mu|^2.$$

$\square$

A *multiplicity function* is a W-invariant map $m : \Sigma \to \mathbb{C}$, $\alpha \mapsto m_\alpha$. We denote the set of multiplicity functions by $\mathcal{M}$. In this article we only consider non-negative multiplicities, i.e. $m_\alpha \geq 0$ for all $\alpha \in \Sigma$. Define

$$\rho = \rho(m) := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha. \tag{2.1}$$

2.3 Definition. Let $\xi \in \mathfrak{a}$ and $m \in \mathcal{M}$. The *Dunkl-Cherednik operator* associated with Σ and m is given by

$$T_\xi = T(\xi, m) := \partial_\xi + \sum_{\alpha \in \Sigma^+} m_\alpha \langle \alpha, \xi \rangle \frac{1}{1 - e^{-2\alpha}} (1 - s_\alpha) - \langle \rho, \xi \rangle,$$

where ∂_ξ is the usual directional derivative and $e^\lambda(\xi) := e^{\langle \lambda, \xi \rangle}$ for $\lambda, \xi \in \mathfrak{a}_{\mathbb{C}}$.

2.4 Remark. Heckman and Opdam use a slightly different notation. They consider a root system R with multiplicity k, which is connected to our notation via

$$R = 2\Sigma, \quad k_{2\alpha} = \frac{1}{2} m_\alpha.$$

Note that this implies further differences. Our notation comes from the theory of symmetric spaces.

For fixed multiplicity m, the operators T_ξ, $\xi \in \mathfrak{a}_\mathbb{C}$ commute. Therefore the assignment $\xi \mapsto T(\xi, m)$ uniquely extends to a homomorphism on the symmetric algebra $S(\mathfrak{a}_\mathbb{C})$ over $\mathfrak{a}_\mathbb{C}$, which may be identified with the algebra of complex polynomials on $\mathfrak{a}_\mathbb{C}$. Let $T(p, m)$ be the operator which corresponds in this way to $p \in S(\mathfrak{a}_\mathbb{C})$. If $p \in S(\mathfrak{a}_\mathbb{C})^W$, the subspace of W-invariant polynomials on $\mathfrak{a}_\mathbb{C}$, then $T(p, m)$ acts as a differential operator on the space of W-invariant analytic functions on $\mathfrak{a}$.

Fix a spectral parameter $\lambda \in \mathfrak{a}_\mathbb{C}$. Then according to fundamental results of Heckman and Opdam (see [6]), the so-called hypergeometric system

$$T(p, m)\varphi = p(\lambda)\varphi \quad \text{for all } \ p \in S(\mathfrak{a}_\mathbb{C})^W$$

has a unique W-invariant solution $\varphi = F_\lambda(m; \cdot) = F(\lambda, m; \cdot)$ which is analytic on $\mathfrak{a}$ and satisfies $F_\lambda(m; 0) = 1$. Moreover, there is a W-invariant tubular neighborhood U of $\mathfrak{a}$ in $\mathfrak{a}_\mathbb{C}$ such that F extends to a (single-valued) holomorphic function $F : \mathfrak{a}_\mathbb{C} \times \mathcal{M}^{\text{reg}} \times U \to \mathbb{C}$. The function $F(\lambda, m; x)$ is W-invariant in both λ and x. It is called the *hypergeometric function* associated with Σ. For certain spectral parameters λ, the functions F_λ are actually trigonometric polynomials, the so-called Heckman-Opdam polynomials. In order to make this precise, we need some more notation.

Let $\mathcal{T} := \text{lin}\{e^{i\lambda} : \lambda \in \Lambda\}$ be the space of trigonometric polynomials associated with Λ. Trigonometric polynomials are $\pi Q^\vee$-periodic, and $T_\xi \mathcal{T} \subset \mathcal{T}$. Consider the torus $T = \mathfrak{a}/\pi Q^\vee$ with the W-invariant weight function

$$w_m := \prod_{\alpha \in \Sigma^+} \left| e^{i\alpha} - e^{-i\alpha} \right|^{m_\alpha}. \tag{2.2}$$

Let

$$M_\lambda := \sum_{\mu \in W.\lambda} e^{i\mu}, \quad \lambda \in \Lambda^+$$

denote the W-invariant orbit sums. They form a basis of the space of W-invariant trigonometric polynomials $\mathcal{T}^W$. For $\lambda \in \Lambda^+$ the *(Heckman-Opdam) Jacobi polynomials* associated with Σ are now defined by

$$P_\lambda = P_\lambda(m; \cdot) := \sum_{\mu \in \Lambda^+,\, \mu \preceq \lambda} c_{\lambda\mu}(m) M_\mu$$

where the coefficients $c_{\lambda\mu}(m)$ are uniquely determined by the conditions

(i) $c_{\lambda\lambda}(m) = 1$

(ii) P_λ is orthogonal to M_μ in $L^2(T; w_m)$ for all $\mu \in \Lambda^+$ with $\mu \prec \lambda$.

The Jacobi polynomials P_λ form an orthogonal basis of $L^2(T, w_m)^W$, the subspace of W-invariant elements from $L^2(T, w_m)$.

2.5 Remark. Notice that our notation slightly differs from that of Heckman and Opdam (e.g. [6], [16]), namely by a factor i in the spectral variable. This choice of notation will be more convenient for our purposes.

The connection between the Jacobi polynomials and the hypergeometric function is as follows:

2.6 Lemma. *(See [6]) For all $x \in \mathfrak{a}_\mathbb{C}$ and $\lambda \in \Lambda^+$,*

$$F_{\lambda+\rho}(m; ix) = c(\lambda + \rho, m) P_\lambda(m; x),$$

where the c-function $c(\lambda + \rho, m) = P_\lambda(m; 0)^{-1}$ is given by

$$c(\lambda + \rho, m) = \prod_{\alpha \in \Sigma^+} \frac{\Gamma(\lambda_\alpha + \rho_\alpha + \frac{1}{4} m_{\alpha/2}) \Gamma(\rho_\alpha + \frac{1}{4} m_{\alpha/2} + \frac{1}{2} m_\alpha)}{\Gamma(\lambda_\alpha + \rho_\alpha + \frac{1}{4} m_{\alpha/2} + \frac{1}{2} m_\alpha) \Gamma(\rho_\alpha + \frac{1}{4} m_{\alpha/2})}.$$

We shall work with a renormalized version of the Jacobi polynomials, defined by

$$R_\lambda(x) := R_\lambda(m; x) := c(\lambda + \rho, m) P_\lambda(m; x). \tag{2.3}$$

They satisfy $R_\lambda(0) = 1$.

Let us identify $\mathfrak{a}$ with $\mathbb{R}^q$. Dividing the torus $T = \mathfrak{a}/\pi Q^\vee$ by the action of the Weyl group W gives the fundamental alcove $\overline{A_0} = \{x \in \mathbb{R}^q : \frac{\pi}{2} \geq x_1 \geq x_2 \geq \ldots \geq x_q \geq 0\}$. We may consider the W-invariant trigonometric polynomials $\mathcal{T}^W$ as functions on the fundamental alcove $\overline{A_0}$. The Jacobi polynomials R_λ are orthogonal with respect to the inner product

$$\langle f, g \rangle_m = \int_{\overline{A_0}} f(x) \overline{g(x)} w_m(x)\, dx,$$

but they are not orthonormal. We put

$$r_\lambda := \frac{1}{\|R_\lambda\|_m^2}. \tag{2.4}$$

3 Compact symmetric spaces and their spherical functions

In this section we recall some basics from the theory of symmetric spaces. Standard references are the books of Helgason [7] and [8].

Let U be a compact semisimple and connected Lie group with Lie algebra $\mathfrak{u}$. Let K be a closed subgroup such that $M := U/K$ is connected and there exists an involutive automorphism $\theta : U \to U$ with $U_0^\theta = K$. Here $U^\theta = \{u \in U : \theta(u) = u\}$ and U_0^θ denotes the identity component of U^θ. Then $M = U/K$ is a *compact symmetric space*.

The derivation of θ gives an involution of the Lie algebra $\mathfrak{u}$, which we also denote by θ. The corresponding Cartan decomposition is given by $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$. Let $\mathfrak{b} \subseteq \mathfrak{q}$ be a maximal abelian subspace. Denote by G the analytic subgroup of the complexification $U_{\mathbb{C}}$ with Lie algebra $\mathfrak{g} := \mathfrak{k} \oplus i\mathfrak{q}$. We have $K \subseteq G$. Put $\mathfrak{p} := i\mathfrak{q}$. Then $\mathfrak{a} := i\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{p}$. Denote by $\theta_{\mathbb{C}}$ the analytic continuation of θ to $U_{\mathbb{C}}$ and let $\tau = \theta|_G$. Then τ is a Cartan involution of G and $K = G^\tau$. The symmetric space G/K is called the *noncompact dual* of U/K. Denote by $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a})$ the restricted root system and by Σ^+ a fixed subset of positive restricted roots.

Recall that for an arbitrary Lie group G with compact subgroup K, a *spherical function* on G is a nonzero, K-biinvariant function $\varphi : G \to \mathbb{C}$ which satisfies the product formula

$$\varphi(g)\varphi(h) = \int_K \varphi(gkh)dk \quad \text{for all } g, h \in G, \tag{3.1}$$

where dk denotes the normalized Haar measure on K.

Assume now that (U, K) is as above, and U is also simply connected. Then the spherical functions on U are indexed by the set

$$\left\{ \nu \in \mathfrak{a} \, : \, \nu_\alpha := \frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \ \ (\forall \alpha \in \Sigma^+) \right\}, \tag{3.2}$$

which coincides with the set $\Lambda^+(U/K)$ of (restrictions of) highest weights of K-spherical irreducible representations of U. Recall that an irreducible unitary representation $\Phi : U \to GL(V)$ is called K- spherical, if there exists a K-fixed vector. More precisely, the following holds ([8], Theorem 3.4, Ch. IV and Theorem 4.1, Ch. V):

3.1 Proposition. *Assume that U is simply connected. Let $\mu \in \Lambda^+(U/K)$ and $\Phi : U \to GL(V)$ be a spherical representation of U. Choose a K-fixed vector $v_\mu \in V$ with $\|v_\mu\| = 1$. Then*

$$\psi_\mu(u) = \langle \Phi(u) v_\mu, v_\mu \rangle$$

is a spherical function on U. Conversely, every spherical function on U is of this form for some unique $\mu \in \Lambda^+(U/K)$.

If U is not simply connected, then the spherical functions on U are in general indexed by a subset of $\{\nu \in \mathfrak{a} \, : \, \nu_\alpha \in \mathbb{Z}^+ \ (\forall \alpha \in \Sigma^+)\}$.

We assume again that U is simply connected, and G/K is the concompact dual of U/K. Then there is a close connection between the spherical functions on G and those on U, which is given by the following important proposition.

3.2 Proposition. *Every spherical function φ_λ on G ($\lambda \in \mathfrak{a}_{\mathbb{C}}$) is an analytic function. It extends to a holomorphic function on $G_{\mathbb{C}}$ if and only if $\lambda = \mu + \rho \in \rho + \Lambda^+$. If it extends we denote the analytic extension also by φ_λ and the* *restriction of this extension to U is a spherical function ψ on U. There is a shift in the index and we have the equality*

$$\varphi_{\mu+\rho}|_U = \psi_\mu, \qquad \mu \in \Lambda^+.$$

Conversely: Every spherical function ψ_μ on U ($\mu \in \Lambda^+$) extends to a holomorphic function ψ_μ on $U_{\mathbb{C}}$ and its restriction to G is a spherical function φ on G. Then $\psi_\mu|_G = \varphi_{\mu+\rho}$.

Proof. Corollary 5.2.3 in [6] and Lemma 2.5 in [3]. □

4 The product formula for Jacobi polynomials

We consider the compact Grassmanians $M = U/K$ where $U = SO(p+q)$, $SU(p+q)$ or $Sp(p+q)$ and $K = SO(p) \times SO(q)$, $S(U(p) \times U(q))$ or $Sp(p) \times Sp(q)$, respectively. We write $U = SU(p+q, \mathbb{F})$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and consider K as a subgroup of $U(p, \mathbb{F}) \times U(q, \mathbb{F})$. We exclude the case $p = q$ and assume $p > q \geq 1$.

4.1 Remark. Note that $SU(p+q, \mathbb{F})$ is simply connected in the case $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, but the group $SO(p+q)$ is not. It has the spin group $Spin(p+q)$ as a double cover. In contrast to the noncompact case some complications can occur in the study of compact symmetric spaces. See e.g. Proposition 1.2, Ch. VII in [7]. Therefore in the study of the real case some care has to be taken. Nevertheless we are able to show (with some technical effort) that the same results hold in the real case.

Now we apply the general theory of Section 3 to the compact symmetric spaces under consideration in this article. We may choose for the maximal abelian subspace $\mathfrak{b} \subset \mathfrak{u}$ the set of all matrices $H_{ix} \in M_{p+q}(\mathbb{F})$ of the form

$$H_{ix} = \begin{pmatrix} & 0_{p\times p} & & i\underline{x} \\ & & & 0_{(p-q)\times q} \\ i\underline{x} & 0_{q\times(p-q)} & & 0_{q\times q} \end{pmatrix},$$

where $\underline{x} := \operatorname{diag}(x_1, \ldots, x_q)$ is the $q \times q$ diagonal matrix corresponding to $x = (x_1, \ldots x_q) \in \mathbb{R}^q$. In the literature often a slightly different identification of $\mathfrak{b}$ without the factor i is used. See eg. [7], p.452. Our choice is dual to $\mathfrak{a} = \{H_x : x \in \mathbb{R}^q\}$. Compare the treatise of the noncompact case in [18]. Naturally, this doesn't change any result. We identify $\mathfrak{a}$ (and its dual space $\mathfrak{a}^*$) with $\mathbb{R}^q$ via $H_x \mapsto x$. The corresponding root system Σ is of type BC_q, with the convention that zero is allowed as a multiplicity. In this way the case B_q which occurs for $\mathbb{F} = \mathbb{R}$ is included. The roots α with their multiplicities m_α are given in the following table. The multiplicities depend on p, q and the real dimension $d = 1, 2, 4$ of $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

root α	multiplicity m_α
$\pm e_i \;\; 1 \leq i \leq q$	$d(p-q)$
$\pm 2e_i, \;\; 1 \leq i \leq q$	$d-1$
$\pm e_i \pm e_j, \;\; 1 \leq i < j \leq q$	d

We will use the notation $m = (m_1, m_2, m_3)$ where m_i $(i = 1, 2, 3)$ denotes the multiplicity on $\pm e_i$, $\pm 2e_i$ or $\pm e_i \pm e_j$, respectively. The canonical choice of positive roots is $\Sigma^+ = \{e_i, 2e_i : 1 \le i \le q\} \cup \{e_i \pm e_j : 1 \le i < j \le q\}$. The Weyl group W, which is generated by the reflections in the hyperplanes

$$H_\alpha = \{x \in \mathbb{R}^q \,:\, \langle \alpha, x\rangle = 0\},$$

is the hyperoctahedral group in all cases. The Weyl chamber associated to Σ^+ is given by

$$\mathfrak{a}^+ = \{H_x \,:\, x = (x_1, \ldots, x_q) \in \mathbb{R}^q \text{ such that } x_1 > x_2 > \ldots > x_q > 0\}.$$

The set of dominant weights is given by

$$\Lambda^+ = \{\lambda \in \mathbb{R}^q \,:\, \lambda_\alpha \in \mathbb{Z}^+ \, (\forall\, \alpha \in \Sigma^+)\} = \begin{cases} (\mathbb{Z}^+)^q & \text{if } \mathbb{F} = \mathbb{R}; \\ 2(\mathbb{Z}^+)^q & \text{if } \mathbb{F} = \mathbb{C}, \mathbb{H}. \end{cases}$$

In the case of a compact symmetric space, the *affine Weyl group* contains more information about the space. For $k \in \pi\mathbb{Z}$ and $\alpha \in \Sigma$ let

$$H_{\alpha,k} = \{x \in \mathbb{R}^q \,:\, \langle \alpha, x\rangle = k\} = H_\alpha + \frac{k}{2}\alpha^\vee$$

and denote by $s_{\alpha,k}$ the reflection in the hyperplane $H_{\alpha,k}$. The affine Weyl group W_{aff} is the (infinite) group generated by the reflections $\{s_{\alpha,k} \,:\, \alpha \in \Sigma,\, k \in \pi\mathbb{Z}\}$. It is a semidirect product of the coroot lattice and the Weyl group $W_{\text{aff}} = Q^\vee \rtimes W$.

The connected components of $\mathbb{R}^q \setminus \{\bigcup_{\alpha\in\Sigma^+, k\in\pi\mathbb{Z}} H_{\alpha,k}\}$ are called alcoves. In the case of BC_q we choose the *fundamental alcove*

$$A_0 = \{x \in \mathbb{R}^q \,:\, \frac{\pi}{2} > x_1 > x_2 > \ldots > x_q > 0\}. \tag{4.1}$$

(The Weyl chamber $\mathfrak{a}^+$ is cut off by $Q^\vee$.) Its closure $\overline{A_0}$ is a fundamental domain for the action of W_{aff} on $\mathfrak{a}$.

The following theorem gives a sharpened KAK-decomposition of U.

4.2 Theorem. *Let $U = SU(p+q, \mathbb{F})$. The group U decomposes as $U = KSK$, where*

$$S = \left\{ b_x = \begin{pmatrix} \cos \underline{x} & 0_{q\times(p-q)} & i\sin \underline{x} \\ 0_{(p-q)\times q} & I_{p-q} & 0_{(p-q)\times q} \\ i \sin \underline{x} & 0_{q\times(p-q)} & \cos \underline{x} \end{pmatrix} \,:\, x \in \overline{A_0} \right\}.$$

Every $u \in U$ can be written as $u = kb_xk'$ with $k, k' \in K$ and a unique $b_x \in S$.

Proof. In the cases $\mathbb{F} = \mathbb{C}, \mathbb{H}$ the group U is simply connected and the result follows from Theorem 8.6. in Chapter VII of [7]: Put $\overline{Q_0} := \{H_{ix} \,:\, x \in \overline{A_0}\}$. Then a short calculation shows that $S = \exp \overline{Q_0}$.

In the case of $SO(p+q)$ we cannot apply this theorem but the decomposition is also valid and unique. See [20], Section 15.1.9. $\square$

Now we turn our attention to the spherical functions on U. Our first aim is to make the product formula

$$\psi(g)\psi(h) = \int_K \psi(gkh)dk \tag{4.2}$$

explicit. For this, we may closely follow the argumentation of [18] (Section 2) in the non-compact dual cases. Since spherical functions on $U = KSK$ are K-biinvariant they are determined by their values on S. We consider

$$g := \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} b_x \begin{pmatrix} \widetilde{u} & 0 \\ 0 & \widetilde{v} \end{pmatrix} \in KSK.$$

and write g in $p \times q$ block notation as

$$g = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}.$$

A short calculation then gives

$$D(g) = v \cos \underline{x}\, \widetilde{v} \tag{4.3}$$

where $\cos \underline{x} = \operatorname{diag}(\cos x_1, \ldots, \cos x_q)$ with $\cos x_i \in [0,1]$ for all i. We denote by $\operatorname{spec}_s(X)$ the *singular spectrum* of $X \in M_q(\mathbb{F})$, that is

$$\operatorname{spec}_s(X) = \sqrt{\operatorname{spec}(X^*X)} = (\sigma_1, \ldots, \sigma_q) \in \underline{\mathbb{R}}^q,$$

with the singular values σ_i of X ordered by size: $\sigma_1 \geq \ldots \geq \sigma_q \geq 0$. Equation (4.3) implies that the singular spectrum of $D(g)$ is given by $\operatorname{spec}_s(D(g)) = (\cos x_1, \ldots, \cos x_q) =: \cos x$. By our choice of the fundamental alcove A_0, we therefore have

$$x = \arccos(\operatorname{spec}_s(D(g)) \quad \forall g \in Kb_xK,\ x \in \overline{A_0}, \tag{4.4}$$

where arccos is also taken componentwise.

In order to evaluate formula (4.2) explicitly, we write $b_x \in S$ in $p \times q$ block notation:

$$b_x = \begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix}.$$

Then for $k = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in K$ we obtain by a short calculation that

$$D(b_xkb_y) = -\sin \underline{x}\, \sigma_0^* u \sigma_0 \sin \underline{y} \,+\, \cos \underline{x}\, v \cos \underline{y}.$$

with the $p \times q$ block matrix

$$\sigma_0 := \begin{pmatrix} I_q \\ 0 \end{pmatrix}.$$

Now let ψ be a spherical function on U and put $\widetilde{\psi}(x) := \psi(b_x)$ for $x \in \overline{A_0}$. From (4.4) it follows that $\widetilde{\psi}$ satisfies

$$\widetilde{\psi}(x)\widetilde{\psi}(y) \;=\; \int_K \widetilde{\psi}\left(\arccos\left(\mathrm{spec}_s(D(b_x k b_y)\right)\right) dk. \tag{4.5}$$

For our later extension of this product formula beyond the geometric cases, it is important to rewrite it in a way where the parameter p is no longer contained in the domain of integration. Under the technical assumption $p \geq 2q$, this can be done in the same way as in [18], which leads to the following

4.3 Proposition. *Suppose that $p \geq 2q$. Define*

$$B_q := \{w \in M_q(\mathbb{F}) : w^*w < I\},$$

$$\gamma := d(q - \frac{1}{2}) + 1,$$

and for $\mu \in \mathbb{C}$ with $\mathrm{Re}\,\mu > \gamma - 1$*, put*

$$\kappa_\mu := \int_{B_q} \Delta(I - w^*w)^{\mu-\gamma} dw$$

where Δ is the usual determinant on $M_q(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and the Dieudonné determinant for $\mathbb{F} = \mathbb{H}$, i.e. $\Delta(X) = (\det_{\mathbb{C}}(X))^{1/2}$, when X is considered as a complex matrix.
Then the spherical functions $\widetilde{\psi}(x) = \psi(b_x)$ $(x \in \overline{A_0})$ on U satisfy the product formula

$$\widetilde{\psi}(x)\widetilde{\psi}(y) \;=\; \frac{1}{\kappa_{pd/2}} \int_{B_q}\int_{U_0(q,\mathbb{F})} \widetilde{\psi}\left(\arccos\left(\mathrm{spec}_s(-\sin\underline{x}\,w\sin\underline{y} \,+\, \cos\underline{x}\,v\cos\underline{y})\right)\right)$$
$$\cdot\, \Delta(I - w^*w)^{pd/2-\gamma} dv dw.$$

The next step is to identify the spherical functions on $SU(p+q,\mathbb{F})$ as Heckman-Opdam Jacobi polynomials of type BC_q.

Consider first the general situation. Let G be a simply connected, semisimple noncompact Lie group with maximal compact subgroup K. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G and K with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and denote by $\Sigma = \Sigma(\mathfrak{g},\mathfrak{a})$ the (restricted) root system with geometric multiplicity m. Then the spherical functions on G are indexed by $\mathfrak{a}_{\mathbb{C}}$, ([8]), where two spherical functions φ_λ and φ_μ coincide iff λ and μ are in the same orbit of the Weyl group $W(\Sigma)$. The following important fact links the theory of Heckman and Opdam with the classical theory of symmetric spaces:

4.4 Proposition. *([6], Theorem 5.2.2) Let φ_λ, $\lambda \in \mathfrak{a}_{\mathbb{C}}$, be a spherical function on G, let Σ and m be the associated restricted root system and geometric multiplicity. Then for $x \in \mathfrak{a}$,*

$$\varphi_\lambda(\exp x) = F_\lambda(m; x).$$

We know from Proposition 3.2 that a spherical function φ_μ on G extends holomorphically to the complexification $G_{\mathbb{C}}$ if and only if $\mu \in \rho + \Lambda^+$. In this case the restriction to the simply connected U is a spherical function ψ_λ on U, where

$$\psi_\lambda = \varphi_{\lambda+\rho}|_U, \quad \lambda \in \Lambda^+.$$

When combining this with Proposition 4.4 and (2.3), we obtain

4.5 Theorem. *Let U/K be a compact symmetric space with dual G/K. Assume that U is simply connected and let $(\mathfrak{a}, \Sigma, m)$ be the associated geometric root data. Then the spherical functions on U - restricted to $i \exp \mathfrak{a}$ - are (Heckman-Opdam) Jacobi polynomials of type Σ and with multiplicity m:*

$$\psi_\lambda(i \exp x) = \varphi_{\lambda+\rho}(i \exp x) = F_{\lambda+\rho}(m; ix) = R_\lambda(m; x), \qquad \text{for all } \lambda \in \Lambda^+,\ x \in \mathfrak{a}.$$

Here we denote by ψ the spherical functions on U and by φ the holomorphic extension of the spherical functions on G to the complexification $G_{\mathbb{C}}$.

The second equality in the theorem above follows from Proposition 4.4 since $\varphi_{\lambda+\rho}$ and $F_{\lambda+\rho}$ extend holomorphically to the complexification.

4.6 The real case $U = SO(p+q)$. As already remarked, the case $U = SO(p+q)$ has to be treated with some care, since this group is not simply connected. If U is not simply connected then we only know that the set $\Lambda^+(U/K)$ of highest weights of K-spherical representations of U is a sublattice of $\{\nu \in \mathfrak{a} : \nu_\alpha \in \mathbb{Z}^+ \ (\forall \alpha \in \Sigma^+)\}$.

The spin group $\widetilde{U} = Spin(p+q)$ is the universal covering group of $U = SO(p+q)$. The spin group is a subset of the Clifford algebra. Denote by $\varphi : \widetilde{U} \to U$ the covering homomorphism. The kernel $S := \ker \varphi = \{\pm 1\}$ is contained in the center of $\widetilde{U}$. Further, there exists a unique involution $\widetilde{\theta} : \widetilde{U} \to \widetilde{U}$ which corresponds to the involution $\theta : \mathfrak{u} \to \mathfrak{u}$ on the Lie algebra. Since $\widetilde{U}$ is simply connected, the group $\widetilde{K} := \widetilde{U}^{\widetilde{\theta}}$ of fixed points of $\widetilde{\theta}$ is connected ([7], Theorem 8.2 in Chapter VII).

We claim that

$$\Lambda^+(U/K) = \Lambda^+(\widetilde{U}/\widetilde{K}) = \left\{\nu \in \mathfrak{a} : \nu_\alpha \in \mathbb{Z}^+ \ (\forall \alpha \in \Sigma^+)\right\}.$$

This is equivalent to showing that every $\widetilde{K}$-spherical representation on $\widetilde{U}$ descends to a K-spherical representation on U. Note that in general if a $\widetilde{K}$-spherical representation descends it will have automatically a K_0-fixed vector, but not necessarily a K-fixed vector (See the remarks in [14] after Theorem 2.1.). Since $K = SO(p) \times SO(q)$ is connected this problem doesn't occur in our case. The fact that every $\widetilde{K}$-spherical representation on $\widetilde{U}$ descends will follow as soon as we know that $S \subset \widetilde{K}$. For this, recall that (U, K) is a Riemannian symmetric pair and that $\theta : U \to U$ is an involution of U which corresponds to the Cartan involution on the Lie algebra $\mathfrak{u}$. Thus Lemma 1.3, Ch. VII, [7] implies $\widetilde{\theta}(S) \subset S$. Since $\widetilde{\theta}$ is an involution we have $\widetilde{\theta}(1) = 1$ and $\widetilde{\theta}(-1) = -1$.

Therefore S is contained in the fixpoint group $\widetilde{K}$.

From this the claim follows either by a remark of Vretare (The spherical function on U coincides with those on $\widetilde{U}$ for which $\varphi(u's) = \varphi(u')$, $u' \in \widetilde{U}$, $s \in S$; see [19], p. 357.) or by the following reasoning: A spherical representation π on $\widetilde{U}$ is irreducible. Since S is contained in the center of $\widetilde{U}$ it acts by a constant (Schur's Lemma). This constant has to be 1 because every K-fixed vector is also S-fixed.

We summarize: The spherical functions on U coincide with those on $\widetilde{U}$.

Using Theorem 4.5 and 4.6 we conclude for the spherical functions ψ_λ ($\lambda \in \Lambda^+$) on our compact Grassmann manifolds U/K over each of the (skew-) fields $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$:

$$\widetilde{\psi}_\lambda(x) = \psi_\lambda(b_x) = F_{BC_q}(\lambda + \rho, m; ix) = R_\lambda(x). \tag{4.6}$$

Under the assumption $p \geq 2q$ we get from Proposition 4.3:

$$R_\lambda(x)R_\lambda(y) = \frac{1}{\kappa_{pd/2}} \cdot \int_{B_q} \int_{U_0(q,\mathbb{F})} R_\lambda\left(d(\underline{x}\,\underline{y}, v, w)\right) \Delta(I - w^*w)^{pd/2-\gamma} dv dw,$$

for all $x, y \in \mathbb{R}^q$, where

$$d(\underline{x}, \underline{y}, v, w) := \arccos\left(\operatorname{spec}_s(-\sin\underline{x}, w \sin\underline{y} + \cos\underline{x}\, v \cos\underline{y})\right). \tag{4.7}$$

The next step is analytic continuation. Fix q and $d = \dim_{\mathbb{R}}\mathbb{F}$. For $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \gamma - 1$ and dominant weight λ define

$$R_\lambda^\mu(x) := F_{BC_q}(\lambda + \rho, m_\mu; ix),$$

where the multiplicity m_μ is given by

$$m_\mu = (2\mu - dq, d - 1, d).$$

For $\mu = pd/2$, with $p \in \mathbb{N}$ we have geometric cases. This means that for these values of μ, the Jacobi polynomials R_λ^μ are the spherical functions of a compact symmetric space U/K.

4.7 Theorem. *For $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \gamma - 1$ as above the Jacobi polynomials R_λ^μ satisfy the product formula*

$$R_\lambda^\mu(x)R_\lambda^\mu(y) = \frac{1}{\kappa_\mu} \int_{B_q} \int_{U_0(q,\mathbb{F})} R_\lambda^\mu\left(d(\underline{x}, \underline{y}, v, w)\right) \Delta(I - w^*w)^{\mu-\gamma} dv dw$$

with $d(\cdots)$ as in (4.7).

Proof. The proof is a direct copy of the first part of the proof of Theorem 4.1 in [18]. Replace the (R, k)-notation by our (Σ, m)-notation and the product formula by our product formula. Then rewrite the claimed formula in terms of $P_\lambda^\mu := \frac{1}{c(\lambda+\rho,m_\mu)} R_\lambda^\mu$ (the standard Heckman-Opdam normalization):

$$P_\lambda^\mu(x)P_\lambda^\mu(y) = \frac{1}{\kappa_\mu \cdot c(\lambda+\rho,m_\mu)} \int_{B_q}\int_{U_0(q,\mathbb{F})} P_\lambda^\mu\left(d(\underline{x},\underline{y},v,w)\right)\Delta(I-w^*w)^{\mu-\gamma}dvdw$$

For fixed $\lambda\in\Lambda^+$, the function $c(\lambda+\rho,m_\mu)$ is bounded away from zero as $\mu\to\infty$ in the half plane $H=\{\mu\in\mathbb{C} : \text{Re } \mu > \gamma-1\}$ (see [18]). Then one uses the fact that the coefficients of the P_λ^μ with respect to the monomial basis $\{e^\nu : \nu\in\Lambda\}$ are rational (see [13], Par. 11), and that the integral

$$\frac{1}{\kappa_\mu}\int_{B_q} |\Delta(I-w^*w)^{\mu-\gamma}|dw$$

converges exactly for Re $\mu>\gamma-1$ and is of polynomial growth as $\mu\to\infty$ in H. This allows to apply Carlson's theorem. For details we refer to [18]. □

5 The rank one case

At this point it is worthwile to spend some time to see how our product formula for Heckman-Opdam Jacobi polynomials generalizes the product formula of classical one-variable Jacobi polynomials for certain indices.

The classical Jacobi polynomials with indices $\alpha,\beta>-1$ are given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\big(\alpha+\beta+n+1,-n,\alpha+1;\frac{1-x}{2}\big) \tag{5.1}$$

where ${}_2F_1$ is the Gaussian hypergeometric function and $(a)_k := a(a+1)\dots(a+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$. We renormalize:

$$R_n^{(\alpha,\beta)}(x) =:= \frac{n!}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(x).$$

Let us consider the Heckman-Opdam theory in the rank one case. The root system is $BC_1 = \{\pm e_1, \pm 2e_1\}$ in $\mathfrak{a}\cong\mathbb{R}$ and we denote the multiplicity by $m_1 := m_\alpha$ and $m_2 := m_{2\alpha}$. Note that zero is allowed as a multiplicity and in fact, we have $m_2=0$ in the real case.

According to the example in [15], p. 89f, the hypergeometric function F_{BC_1} is given by

$$F_{BC_1}(\lambda,m;x) = {}_2F_1\big(a,b,c;\frac{1}{2}(1-\cosh 2x)\big). \tag{5.2}$$

Here

$$a=\frac{1}{2}\big(\lambda+\frac{1}{2}m_1+m_2\big),\quad b=\frac{1}{2}\big(-\lambda+\frac{1}{2}m_1+m_2\big) \text{ and } c=\frac{1}{2}\big(1+m_1+m_2\big). \tag{5.3}$$

The dominants weights in the rank one case are

$$\Lambda^+(\Sigma) = \begin{cases} \mathbb{Z}^+ & \text{if } \mathbb{F}=\mathbb{R} \\ 2\mathbb{Z}^+ & \text{if } \mathbb{F}=\mathbb{C},\mathbb{H}. \end{cases}$$

According to Lemma 2.6 we have

$$F_{\lambda+\rho}(ix) = R_\lambda(x).$$

In the case $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$ let $\mu = \lambda+\rho = 2n+\frac{1}{2}m_1+m_2 \in \rho+2\mathbb{Z}^+$ and choose a, b, c as in (5.3). Then we get from equation (5.2)

$$F_{BC_1}(\mu, m; x) = {}_2F_1\big(n+\frac{1}{2}m_1+m_2, -n, \frac{1}{2}(1+m_1+m_2); \frac{1}{2}(1-\cosh 2x)\big)$$

and with equation (5.1) we conclude

$$R_\lambda(x) = R_n^{(\alpha,\beta)}(\cos 2x)$$

where

$$\alpha = \frac{1}{2}(m_1+m_2-1), \quad \beta = \frac{1}{2}(m_2-1).$$

In the real case $\mathbb{F} = \mathbb{R}$ the rank one symmetric space $SO(p+1)/SO(p)$ is the sphere $\mathbb{S}^p$, the root system $\Sigma = \{\pm e_1\}$ is of type B_1 and the dominant weights are the natural numbers. It is well known that the spherical functions of $\mathbb{S}^p$ are Gegenbauer polynomials. In fact, we have in the real rank one case for a weight $\lambda = n \in \mathbb{Z}^+$

$$R_\lambda(x) = F_{B_1}(\lambda+\rho, m; ix) = {}_2F_1\left(\frac{n}{2}+\frac{1}{2}m_1, -\frac{n}{2}, \frac{1}{2}(1+m_1); \frac{1}{2}(1-\cos 2x)\right).$$

Using the hypergeometric identity (3.1.3) in [1], we obtain

$$\begin{aligned} R_\lambda(x) &= {}_2F_1\left(n+m_1, -n, \frac{1}{2}(1+m_1); \frac{1-\cos x}{2}\right) \\ &= R_n^{(\alpha,\alpha)}(\cos x), \end{aligned}$$

where $\alpha = \frac{1}{2}(m_1-1)$. So the Heckman-Opdam Jacobi polynomials on the sphere are indeed Gegenbauer polynomials.

To summarize: The spherical functions of the compact rank one symmetric space U/K are given by (classical) Jacobi polynomials. This is well known, see e.g. Theorem 4.5, Ch. V in [8].

In the 1970ies, Koornwinder devoted a series of papers to the product formula for one-variable Jacobi polynomials, see e.g. [12]. For arbitrary $\alpha > \beta > -\frac{1}{2}$, it is given by

$$\begin{aligned} R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) = \int_0^1\int_0^\pi R_n^{(\alpha,\beta)}\big(&\frac{1}{2}(1+x)(1+y)+\frac{1}{2}(1-x)(1-y)r^2 \\ &+\sqrt{1-x^2}\sqrt{1-y^2}r\cos\theta - 1\big)\, dm_{\alpha,\beta}(r,\theta) \end{aligned}$$

with

$$dm_{\alpha,\beta}(r,\theta) = c_{\alpha,\beta}(1-r^2)^{\alpha-\beta-1}(r\sin\theta)^{2\beta} r\, dr d\theta$$

and

$$\frac{1}{c_{\alpha,\beta}} = \int_0^1 \int_0^\pi (1-r^2)^{\alpha-\beta-1}(r\sin\theta)^{2\beta} r\,dr d\theta.$$

Now consider the product formula from Theorem 4.7 in the geometric cases for rank $q=1$. The geometric multiplicities are

$$m_1 = d(p-1) \text{ and } m_2 = d-1,$$

where $d = \dim_{\mathbb{R}}\mathbb{F} \in \{1,2,4\}$ and $p \in \mathbb{N}$, $p \geq 2$. Then we use the identification (5) with parameters

$$\alpha = \frac{1}{2}(m_1+m_2-1) = \frac{1}{2}dp - 1 \text{ and } \beta = \frac{1}{2}(m_2-1) = \frac{1}{2}d - 1.$$

The domains of integration reduce to $B_1 = \{w \in \mathbb{F} : |w| < 1\}$ and $U_0(1) = \{v \in \mathbb{F} : |v| = 1\}_0$. Furthermore we have

$$d(x,y,v,w) = \arccos|-w\sin x\sin y + v\cos x\cos y|$$

and $\Delta(I-w^*w)^{\mu-\gamma} = (1-|w|^2)^{\frac{d}{2}(p-1)-1}$. The $U_0(1)$-integral cancels under the coordinate transform $w' := v^{-1}w$. Using $\cos 2x = 2\cos^2 x - 1$ we obtain

$$R_n^{(\alpha,\beta)}(\cos 2x)R_n^{(\alpha,\beta)}(\cos 2y) = \frac{1}{\kappa_{pd/2}}\int_{B_1} R_n^{(\alpha,\beta)}(2|-z\sin x\sin y + \cos x\cos y|^2 - 1)\cdot(1-|z|^2)^{\frac{d}{2}(p-1)-1}dz. \tag{5.4}$$

Let us sketch the further calculations only in the case $\mathbb{F} = \mathbb{C}$. We introduce polar coordinates $z = re^{i\theta}$ and put $t := \cos 2x$, $s := \cos 2y$. Then use the identities $\sin^2 x = \frac{1}{2}(1-t)$, $\sin x\cos x = \frac{1}{2}\sqrt{1-t^2}$ and $\cos^2 x = \frac{1}{2}(1+t)$. The constant $\kappa_{pd/2}$ is given by

$$\kappa_{pd/2} = 2\pi\int_0^1 (1-r^2)^{p-2} r\,dr = \frac{\pi}{p-1}.$$

We conclude from (5.4) exactly the product formula for Jacobi polynomials with $\alpha = p-1$ and $\beta = 0$:

$$\begin{aligned} R_n^{(p-1,0)}(t)R_n^{(p-1,0)}(s) = &\frac{2(p-1)}{\pi}\int_0^1\int_0^\pi R_n^{(p-1,0)}(\frac{1}{2}(1+t)(1+s) + \frac{1}{2}(1-t)(1-s)r^2 \\ &+ \sqrt{1-t^2}\sqrt{1-s^2}r\cos\theta - 1)(1-r^2)^{p-2} r\,dr d\theta. \end{aligned}$$

6 Hypergroup structures on the alcove

In this section we shall see that the product formula of Theorem 4.7 leads to three continuous series (for $d = 1,2,4$) of positivity-preserving convolution algebras on the fundamental alcove $\overline{A_0} = \{t \in \mathbb{R}^q : \frac{\pi}{2} \geq t_1 \geq \ldots \geq t_q \geq 0\}$, which are compact commutative hypergroups with normalized Jacobi polynomials as characters. In the geometric cases ($\mu = pd/2$), these hypergroups convolutions are just given by the double coset convolutions on the double coset space $U//K$ which may be identified with $\overline{A_0}$ by Theorem 4.2.

To start with, let us briefly recall some basics from hypergroup theory. For a detailed treatment, the reader is referred to [11]. Hypergroups generalize the convolution algebras of locally compact groups, with the convolution product of two point measures δ_x and δ_y being in general not a point measure again but a probability measure with compact support depending on x and y.

6.1 Definition. A *hypergroup* is a locally compact Hausdorff space X with a weakly continuous, associative convolution $*$ on the space $M_b(X)$ of regular bounded Borel measures on X, satisfying the following properties:

(i) The convolution product $\delta_x * \delta_y$ of two point measures is a compactly supported probability measure on X, and $\text{supp}(\delta_x * \delta_y)$ depends continuously on x and y with respect to the so-called Michael topology on the space of compact subsets of X (see [11]).

(ii) There exists a (necessarily unique) neutral element $e \in X$ satisfying $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in X$.

(iii) There exists a (necessarily unique) continuous involution $x \mapsto \overline{x}$ on X such that $\delta_{\overline{x}} * \delta_{\overline{y}} = (\delta_y * \delta_x)^-$ and $x = \overline{y} \iff e \in \text{supp}(\delta_x * \delta_y)$. (Here the measure μ^- is given by $\mu^-(A) = \mu(\overline{A})$.)

The hypergroup is called *commutative* if the convolution is commutative.

Note that due to weak continuity, the convolution of measures on a hypergroup is uniquely determined by the convolution of point measures.

Every commutative hypergroup X has a unique (up to a multiplicative factor) *Haar measure* ω, that is a positive Radon measure with the property

$$\int_X f(x*y)d\omega(y) = \int_X f(y)d\omega(y) \quad (\forall x \in X, f \in C_c(X)),$$

where we use the notation $f(x*y) := (\delta_x * \delta_y)(f)$.

The *dual space* of a hypergroup X is defined by

$$\widehat{X} := \{\varphi \in C_b(X) \,:\, \varphi \neq 0,\ \varphi(\overline{x}) = \overline{\varphi(x)} \text{ and } \varphi(x*y) = \varphi(x)\varphi(y)\}.$$

The elements of $\widehat{X}$ are called *characters* of X. As in the case of locally compact abelian groups, the dual of a commutative hypergroup is a locally compact Hausdorff space with the topology of locally uniform convergence. In general the dual is not again a hypergroup. In the case of a compact hypergroup X the dual $\widehat{X}$ is discrete. The Fourier transform on $L^1(X,\omega)$ is defined by $\widehat{f}(\varphi) := \int_X f(x)\overline{\varphi(x)}d\omega(x)$. It is injective and there exists a unique positive Radon measure π on $\widehat{X}$, called the *Plancherel measure* of $(X,*)$, such that $f \mapsto \widehat{f}$ extends to an isometric isomorphism from $L^2(X,\omega)$ onto $L^2(\widehat{X},\pi)$.

6.2 Example. (Double coset hypergroups)
Let G be a locally compact group with compact subgroup K and denote by dk the Haar measure on K. Then there is a natural hypergroup structure on the set of double cosets $G//K = \{KxK : x \in G\}$ which is given by

$$\delta_{KxK} * \delta_{KyK} = \int_K \delta_{KxkyK}\, dk, \quad x, y \in G.$$

The neutral element is $K = KeK$ and the involution is given by $(KxK)^- = Kx^{-1}K$ (see Theorem 8.2B in [11]). The double coset hypergroup $(G//K, *)$ is commutative if and only if (G, K) is a Gelfand pair.

6.3 Theorem. *Let $\mu \in \mathbb{C}$ with $\mathrm{Re}\ \mu > \gamma - 1$. Then the probability measures*

$$\begin{aligned}\delta_x *_\mu \delta_y(f) :=& \\ &\frac{1}{\kappa_\mu}\int_{B_q}\int_{U_0(q,\mathbb{F})} f\left(\arccos\left(\mathrm{spec}_s(-\sin \underline{x}\, w \sin \underline{y} + \cos \underline{x}\, v \cos \underline{y})\right)\right) \cdot \\ &\cdot \Delta(I - w^*w)^{\mu-\gamma} dv dw\end{aligned}$$

for $x, y \in \overline{A_0}$ define a commutative hypergroup structure on the compact alcove $\overline{A_0}$. The neutral element is 0 and the involution is the identity mapping.

Note that in the geometric cases $\mu = pd/2$ the hypergroup on $\overline{A_0}$ coincides with the double coset hypergroup $U//K$.

Proof. Commutativity is obvious. For associativity let $x, y, z \in \overline{A_0}$. Then

$$\begin{aligned}\delta_x *_\mu (\delta_y *_\mu \delta_z)(f) =& \frac{1}{\kappa_\mu^2}\int_{B_q \times U_0(q)}\int_{B_q \times U_0(q)} f(D(x, y, z, v, w, v', w'))\cdot \\ &\cdot \Delta(I - w^*w)^{\mu-\gamma}\Delta(I - (w')^*w')^{\mu-\gamma} dv dw dv' dw' \quad =: I(\mu)\end{aligned}$$

with a certain $\overline{A_0}$ valued argument D, which is independent of μ. The same is true for

$$(\delta_x *_\mu \delta_y) *_\mu \delta_z(f) =: I'(\mu)$$

with a μ-independent argument D' instead of D. The integrals $I(\mu)$ and $I'(\mu)$ are well defined and holomorphic in $\{\mu \in \mathbb{C} : \mathrm{Re}\,\mu > \gamma - 1\}$. The convolution is associative in the geometric cases $\mu = pd/2$. Analytic continuation yields associativity for all μ with $\mathrm{Re}\,\mu > \gamma - 1$ as in [17]. Weak continuity of the convolution follows from the continuity of the mapping $(x, y, v, w) \mapsto f(d(x, y, v, w))$ on $\overline{A_0}^2 \times B_q \times U_0(q, \mathbb{F})$. The compact support of $\delta_x * \delta_y$ is trivial. It is also obvious that 0 is neutral. So only the support continuity with respect to the Michael topology and the fact that the identity mapping is a hypergroup involution remain. As the support of $\delta_x *_\mu \delta_y$ is independent of μ, it suffices to verify both statements in the geometric cases $U//K$. But these are known to correspond to double coset hypergroups, which immediately implies the support continuity. In the geometric cases, the involution is induced from the mapping

$x \mapsto -x$ on $\mathbb{R}^q \cong \mathfrak{a}$. A short calculation shows that $b_{-x} \in K b_x K$ on $U//K$ and therefore the involution on $U//K$ is the identity. In fact,

$$\begin{pmatrix} \cos \underline{x} & & i \sin \underline{x} \\ & I_{p-q} & \\ i \sin \underline{x} & & \cos \underline{x} \end{pmatrix} =$$

$$= \begin{pmatrix} -I_q & & \\ & I_{p-2q,q} & \\ & & I_q \end{pmatrix} \begin{pmatrix} \cos \underline{x} & & -i \sin \underline{x} \\ & I_{p-q} & \\ -i \sin \underline{x} & & \cos \underline{x} \end{pmatrix} \begin{pmatrix} -I_q & & \\ & I_{p-2q,q} & \\ & & I_q \end{pmatrix}$$

where $I_{n,m} = \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1)$ denotes the diagonal matrix with the first n entries equal to 1 and the last m entries equal to -1. Then
$\det \begin{pmatrix} -I_q & \\ & I_{p-2q,q} \end{pmatrix} = 1.$ □

6.4 Proposition. *The support of $\delta_x *_\mu \delta_y$ satisfies*

$$\operatorname{supp}(\delta_x *_\mu \delta_y) \subseteq \{z \in \overline{A_0} \,:\, \|z\|_\infty \leq \|x\|_\infty + \|y\|_\infty\}$$

where $\|\cdot\|_\infty$ is the maximum norm in $\mathbb{R}^q$.

Proof. For a matrix $A \in M_q(\mathbb{F})$ we denote by

$$\sqrt{\operatorname{spec}(A^* A)} = \operatorname{spec}_s(A) = \{\sigma_i(A) \,:\, i = 1, \ldots, q\}$$

the singular values of A, decreasingly ordered by size: $\sigma_1(A) \geq \ldots \geq \sigma_q(A) \geq 0$. Write $\|A\| = \|\operatorname{spec}_s(A)\|_\infty = \sigma_1(A)$ for the spectral norm of A. Recall the following estimates from Theorem 3.3.16 in [9]:

$$|\sigma_q(A + B) - \sigma_q(A)| \leq \sigma_1(B) \tag{6.1}$$

$$\sigma_q(AB) \leq \sigma_q(A)\sigma_1(B) \tag{6.2}$$

These estimates are only given for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ but in the case of a quaternionic matrix $A \in M_q(\mathbb{H})$ we simply consider the corresponding complex matrix $\chi_A \in M_{2q}(\mathbb{C})$, namely

$$\chi_A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}$$

where A_1, A_2 are complex $q \times q$-matrices such that $A = A_1 + A_2 j$. The map $M_q(\mathbb{H}) \to M_{2q}(\mathbb{C})$, $A \mapsto \chi_A$ is a homomorphism and $\chi_{A^*} = (\chi_A)^*$. Moreover, $\operatorname{spec}_s(A) = \operatorname{spec}_s(\chi_A)$ where in the second set each singular value appears twice (see [21] for a survey about quaternionic matrices).

Let $\xi := \cos \underline{x}\, v\, \cos \underline{y} - \sin \underline{x}\, w\, \sin \underline{y}$. By (6.1),

$$\sigma_q(\xi) \geq \sigma_q(\cos \underline{x}\, v\, \cos \underline{y}) - \sigma_1(\sin \underline{x}\, w\, \sin \underline{y}).$$

Since $\sin x$ is increasing on $[0, \pi/2]$ we get (using submultiplicativity)

$$\sigma_1(\sin \underline{x}\, w\, \sin \underline{y}) = \|\sin \underline{x}\, w\, \sin \underline{y}\| \leq \|\sin \underline{x}\| \|\sin \underline{y}\| = \sin \|x\|_\infty \sin \|y\|_\infty.$$

On the other hand, if $y_i \neq 0$ for all i, then by (6.2)

$$\sigma_q(\cos \underline{x}\, v\, \cos \underline{y}) \geq \frac{\sigma_q(\cos \underline{x}\, v)}{\sigma_1\left((\cos \underline{y})^{-1}\right)} \geq \cos \|x\|_\infty \cos \|y\|_\infty .$$

Therefore

$$\sigma_q(\xi) \geq \cos \|x\|_\infty \cos \|y\|_\infty - \sin \|x\|_\infty \sin \|y\|_\infty = \cos(\|x\|_\infty + \|y\|_\infty)$$

This implies the claim, because arccos is decreasing. If $y_i = 0$ for some i, the estimate remains by continuity since the eigenvalues of a matrix depend continuously upon its entries; see e.g. [9], p. 396.)

□

Because of Theorem 4.7 the Jacobi polynomials R_λ are multiplicative,

$$R_\lambda(x)R_\lambda(y) = R_\lambda(x *_\mu y).$$

6.5 Lemma. *Assume that the Weyl group W contains the reflection $\sigma : x \mapsto -x$. Then the Jacobi polynomials R_λ are real-valued on $\mathbb{R}^q$. In particular, this holds for the root systems B_q, C_q and BC_q.*

Proof. The polynomial R_λ is a linear combination of the orbit sums M_μ, where

$$M_\mu = \sum_{\gamma \in W.\mu} e^{i\gamma}.$$

Since $\sigma \in W$ we sum up $e^{i\gamma_1} + e^{-i\gamma_1} + \ldots$. This is real-valued. □

Therefore the R_λ, $\lambda \in \Lambda^+$ are characters of the hypergroup $(\overline{A_0}, *)$. In fact they are all:

6.6 Proposition. *(a) The Haar measure of the hypergroup $(\overline{A_0}, *_\mu)$ is given by*

$$d\omega(x) = w_m(x)dx = \prod_{\alpha \in \Sigma^+} \left| e^{i\langle \alpha, x\rangle} - e^{-i\langle \alpha, x\rangle} \right|^{m_\alpha} dx.$$

(b) The dual space is $(\overline{A_0}, *_\mu)^\wedge = \{R_\lambda : \lambda \in \Lambda^+\}$.

Proof. (a) For a Jacobi polynomial R_λ with $\lambda \neq 0$ we have $\int_{\overline{A_0}} R_\lambda d\omega = 0$ since R_λ is orthogonal to $R_0 = 1$ with respect to $\langle \cdot, \cdot \rangle_m$. For $z \in \overline{A_0}$ consider the generalized translation

$$(\tau_z f)(x) := f(z *_\mu x) = \int_{\overline{A_0}} f(y) d(\delta_z *_\mu \delta_x)(y).$$

In view of (6), we obtain

$$\int_{\overline{A_0}} (\tau_z R_\lambda)(x) d\omega(x) = \int_{\overline{A_0}} R_\lambda(z *_\mu x) d\omega(x) = R_\lambda(z) \int_{\overline{A_0}} R_\lambda(x) d\omega(x) = 0.$$

By linearity, the above equation holds for all W-invariant trigonometric polynomials. Note that $\mathcal{T}^W$ is dense in $C(\overline{A_0})$ with respect to the norm $\|\cdot\|_\infty$ (Stone-Weierstrass). Now the assertion follows from the $\|\cdot\|_\infty$-continuity of the generalized translation (see Lemma 3.3B in [11]).

(b) We already know that the R_λ are characters of our hypergroup. In general, the characters of a compact hypergroup X form an orthogonal basis of $L^2(X, d\omega)$ (The proof is the same as in the case of a compact group and uses the Plancherel Theorem. See Theorem 3.5 in [5]). The Jacobi polynomials form already an orthogonal basis of $L^2(\overline{A_0})$. So there are no additional characters. $\square$

6.7 Remark. For a general commutative hypergroup X the set of bounded semi-characters

$$\chi_b(X) := \{\varphi \in C_b(X) \,:\, \varphi \neq 0 \text{ and } \varphi(x * y) = \varphi(x)\varphi(y)\}.$$

may not coincide with the dual $\widehat{X}$. However, in the case of a compact commutative hypergroup X (or more general a commutative hypergroup of subexponential growth), one has $\widehat{X} = \chi_b(X)$; see Theorem 2.5.12 in [2]. But Lemma 6.5 is much simpler and of some interest on its own.

We identify the dual of our hypergroup with the set of dominant weights via the mapping $(\overline{A_0})^\wedge \to \Lambda^+$, $R_\lambda \mapsto \lambda$.

6.8 Proposition. *The Plancherel measure of the hypergroup* $(\overline{A_0}, *_\mu)$ *is the following measure on* Λ^+*:*

$$\pi = \sum_{\nu \in \Lambda^+} r_\nu \delta_\nu,$$

where δ_ν *denotes the point measure in* $\nu \in \Lambda^+$*.*

Proof. The set $\{r_\lambda^{1/2} R_\nu \,:\, \nu \in \Lambda^+\}$ is an orthonormal basis of $L^2(\overline{A_0}, \omega)$. Thus for $f \in L^2(\overline{A_0}, \omega)$ we obtain

$$\int_{\overline{A_0}} |f|^2 d\omega = \sum_{\nu \in \Lambda^+} r_\nu | < f, R_\nu > |^2 = \sum_{\nu \in \Lambda^+} r_\nu |\widehat{f}(\nu)|^2 = \int_{\Lambda^+} |\widehat{f}|^2 d\pi.$$

$\square$

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