TU Clausthal

Mathematik-Bericht 2010/3

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April 2010

L^p-APPROXIMATION OF THE INTEGRATED DENSITY OF STATES FOR SCHRÖDINGER OPERATORS WITH FINITE LOCAL COMPLEXITY

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ABSTRACT. We study spectral properties of Schrödinger operators on $\mathbb{R}^d$. The electromagnetic potential is assumed to be determined locally by a colouring of the lattice points in $\mathbb{Z}^d$, with the property that frequencies of finite patterns are well defined. We prove that the integrated density of states (spectral distribution function) is approximated by its finite volume analogues, i.e. the normalised eigenvalue counting functions. The convergence holds in the space $L^p(I)$ where I is any finite energy interval and $1 \le p < \infty$ is arbitrary.

1. INTRODUCTION

Spectral properties play a key role in the analysis of selfadjoint operators. This is in particular the case for Hamiltonians describing the time evolution of quantum mechanical systems. Among the various features of the spectrum which may be studied, a prominent role is played by the Lebesgue decomposition of the components of the spectral measure. A similarly important role is played by the spectral density. To be more precise: there are many selfadjoint operators which have a well defined spectral distribution function. It describes the concentration of the spectrum along the energy axis. In the context of mathematical physics it is called integrated density of states, in the following abbreviated by IDS. It is very natural to think of the IDS as the normalized eigenvalue counting function of the restriction of the Hamiltonian to a large but finite volume system. This leads to the question in what sense and how well one can approximate the IDS of the full Hamilton operator by the spectral distribution functions of appropriately chosen finite-volume analogues. This question has been pursued for various types of selfadjoint operators, resp. Hamiltonians, in particular in the mathematical physics and geometry literature. Let us mention the seminal papers [Pas71, Shu79] and the recent reviews [KM07, Ves08]. There one can find also an overview of the literature up to '07.

It turns out that in the discrete and in the one-dimensional setting one can control the convergence of finite volume approximants to the IDS very well. Let us state this more precisely:

- For difference operators (with finite range) on combinatorial graphs the eigenvalue counting functions are bounded. This leads to uniform convergence (in supremum norm) for the IDS [LMV08, LV09].
- For Schrödinger operators on metric graphs (so called quantum graphs) with constant edge lengths one can achieve uniform convergence for the IDS as well [GLV07]. Here one has to assume that the randomness satisfies a finite local complexity condition. Note that for quantum graphs the eigenvalue counting functions are unbounded: However, the technically relevant objects are the spectral shift functions, which are still bounded.
- Metric graphs with non-constant edge lengths lead to unbounded shift functions; in this case, convergence holds locally uniformly, as well as globally uniformly with respect to a weighted supremum norm [GLV08].

See also [GLV08] for an overview.

For electromagnetic Schrödinger operators on $\mathbb{R}^d$ even the perturbation by a compactly supported potential may lead to a locally unbounded spectral shift function. Thus the shift function diverges not only at infinity but also on compact energy intervals. This is in particular the case for Landau-type Hamiltonians, see, e.g., [RW02, HKN$^+$06] and references therein. One may expect that the situation will be better for certain random perturbations of the Landau Hamiltonian. In order to obtain continuity of the IDS of Landau-type Hamiltonians plus a random, ergodic potential one has to pose appropriate conditions on the randomness. They amount to regularity conditions on the random distribution (see [CH96, Wan97, HLMW01, CHK07] and references therein). The results of the present paper apply to highly "singular" distributions, though: those of Bernoulli type. For such models there are no results on the continuity of the IDS. Thus this property cannot help us proving uniform convergence of the distribution functions. Our main result is that one can still achieve a strong form of convergence. More precisely, we show that convergence holds in the space $L^p(I)$ for any finite interval $I \subset \mathbb{R}$ and any finite p.

In the following section we describe our model and assumptions and state the main theorems. Section 3 provides bounds on the spectral shift function. They are applied in Section 4 which establishes certain almost additivity properties and thus concludes the proof of the main theorem. The latter is applied to certain types of alloy type random Schrödinger operators in the final section.

2. Model and results

Throughout the paper we will consider electromagnetic Schrödinger operators which satisfy the following regularity

Assumption 1. *Let $\mathcal{U}$ be an open set in $\mathbb{R}^d$, $A\colon \mathcal{U} \to \mathbb{R}^d$ a magnetic vector potential, each component of which is locally square integrable, $V = V_+ - V_-\colon \mathcal{U} \to \mathbb{R}$ a scalar electric potential, such that its positive part $V_+ \geq 0$ is locally integrable and its negative part $V_- \geq 0$ is in the Kato class. This implies that V_- is relatively form bounded with respect to $-\Delta^{\mathcal{U}}$, the Dirichlet*

Laplacian on $\mathcal{U}$, with relative bound δ strictly smaller than one. Under these conditions the magnetic Schrödinger operator

$$(1) \qquad\qquad H^{\mathcal{U}} = (-i\nabla - A)^2 + V$$

is well defined via the corresponding lower semi-bounded quadratic form with core $C_c^\infty(\mathcal{U})$ [Sim79]. Due to this choice of core, we say that $H^{\mathcal{U}}$ has Dirichlet boundary conditions.

Let us mention locally uniform L^p-integrability conditions which are sufficient for V_- to be in the Kato-class. More precisely, if V_- satisfies

$$\|V_-\|_{L^p_{\mathrm{loc,unif}}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} \Big(\int_{|x-y| \leq 1} |V_-(y)|^p \, dy \Big)^{1/p} < \infty$$

for $p = 1$ if $d = 1$ and $p > d/2$ if $d \geq 2$, then it belongs to the Kato-class.

Next we want to introduce the notions of a colouring and a pattern. For this purpose we denote the set of all finite subsets of $\mathbb{Z}^d$ by $\mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d)$ and an arbitrary finite set by $\mathcal{A}$. A *colouring* is a map

$$\mathcal{C} : \mathbb{Z}^d \to \mathcal{A}$$

and a *pattern* is a map $P : D(P) \to \mathcal{A}$, where $D(P) \in \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d)$ is called the domain of P. We denote the *set of all patterns* by $\mathcal{P}$. For a fixed $Q \in \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d)$ we denote the subset of $\mathcal{P}$ which contains only the patterns with domain Q by $\mathcal{P}(Q)$. Given a set $Q \subset D(P)$ and an element $x \in \mathbb{Z}^d$ we define a *restriction of a pattern* $P|_Q$ and the *translate of a pattern* $P + x$ by the vector x in the following way

$$P|_Q : Q \to \mathcal{A}, g \mapsto P|_Q(g) = P(g), \quad P + x : D(P) + x \to \mathcal{A}, y + x \mapsto P(y)$$

Two patterns P_1, P_2 are *equivalent* if there exists an $x \in \mathbb{Z}^d$ such that $P_2 = P_1 + x$. The equivalence class of a pattern P in $\mathcal{P}$ is denoted by $\tilde{P}$. This induces on $\mathcal{P}$ a set of equivalence classes $\tilde{\mathcal{P}}$. For two patterns P and P' the number of occurrences of the pattern P in P' is denoted by

$$\sharp_P(P') := \sharp\{x \in \mathbb{Z}^d \mid D(P) + x \subset D(P'), P'|_{D(P)+x} = P + x\}.$$

Here, $\sharp$ denotes the cardinality of a finite set.

Next we define the notion of a van Hove sequence and of the frequency of a pattern along a given van Hove sequence. A sequence $(U_j)_{j \in \mathbb{N}}$ of finite, non-empty subsets of $\mathbb{Z}^d$ is called a van Hove sequence if

$$(2) \qquad\qquad \text{for all } M \in \mathbb{N} : \quad \lim_{j \to \infty} \frac{\sharp \partial^M U_j}{\sharp U_j} = 0.$$

Here $\partial^M U = \{x \in U \mid \mathrm{dist}(x, \mathbb{Z}^d \setminus U) \leq M\} \cup \{x \in \mathbb{Z}^d \setminus U \mid \mathrm{dist}(x, U) \leq M\}$.

It is sufficient to check the relation (2) for $M = 1$, it then follows for all $M \in \mathbb{N}$, cf. for instance Lemma 2.1 in [LSV10]. If for a pattern P and a van Hove sequence $(U_j)_{j \in \mathbb{N}}$ the limit

$$\nu_P := \lim_{j \to \infty} \frac{\sharp_P(\mathcal{C}|_{U_j})}{\sharp U_j}.$$

exists, we call ν_P the *frequency of P along* $(U_j)_{j \in \mathbb{N}}$ in the colouring $\mathcal{C}$.

In our setting $\mathcal{A}$ will be a finite collection of pairs (a, v) where a is a function $\mathbb{R}^d \to \mathbb{R}^d$ such that all its components are in L^2 and v is a function $\mathbb{R}^d \to \mathbb{R}$ such that its positive part is in L^1 and its negative part is in the Kato class, cf. Assumption 1. Moreover, both the support of a and of v are contained in

$$W_0 := [0, 1]^d.$$

Given a colouring $\mathcal{C} \colon \mathbb{Z}^d \to \mathcal{A}$ we denote by $\mathcal{C}_a$ its first and by $\mathcal{C}_v$ its second component. To each colouring $\mathcal{C}$ we associate an electromagnetic potential $(A_\mathcal{C}, V_\mathcal{C}) \colon \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}$

$$A_\mathcal{C}(x) := \sum_{k \in \mathbb{Z}^d} \mathcal{C}_a(k)(x - k), \quad V_\mathcal{C}(x) := \sum_{k \in \mathbb{Z}^d} \mathcal{C}_v(k)(x - k)$$

and a Schrödinger operator

$$H_\mathcal{C} := (-i\nabla - A_\mathcal{C})^2 + V_\mathcal{C}.$$

Note that for any open subset $\mathcal{U}$ of $\mathbb{R}^d$ the restriction $H_\mathcal{C}^\mathcal{U}$ of $H_\mathcal{C}$ to $\mathcal{U}$ with Dirichlet boundary conditions satisfies Assumption 1.

Now we want to define the IDS of $H_\mathcal{C}$ and of finite restrictions thereof. For this purpose we need some more notation. In order to associate finite subsets of $\mathbb{Z}^d$ with bounded subsets of $\mathbb{R}^d$, we define

$$W \colon \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d) \to \mathcal{B}(\mathbb{R}^d), \quad Q \mapsto W_Q := \bigcup_{t \in Q} (W_0 + t)$$

where we use the natural embedding $\mathbb{Z}^d \subset \mathbb{R}^d$ and denote the Borel-σ-algebra by $\mathcal{B}$. Furthermore, if $\mathcal{U}$ is an open set in $\mathbb{R}^d$, $H^\mathcal{U}$ a Schrödinger operator defined on $\mathcal{U}$ and $Q \in \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d)$ such that $\overset{\circ}{W}_Q \subset \mathcal{U}$ we define, by a slight but natural abuse of notation,

$$H^Q := H^{\overset{\circ}{W}_Q},$$

i.e. the restriction of $H^\mathcal{U}$ to the interior of W_Q in the sense of quadratic forms as above.

Let us denote by $\chi_{(-\infty, \lambda]}(H_\mathcal{C})$ and $\chi_{(-\infty, \lambda]}(H_\mathcal{C}^Q)$ the spectral families of $H_\mathcal{C}$ and $H_\mathcal{C}^Q$, respectively. The IDS of $H_\mathcal{C}^Q$ is the distribution function

$$N(\lambda, Q) := \mathrm{Tr}\big[\chi_{(-\infty, \lambda]}(H_\mathcal{C}^Q)\big],$$

divided by the volume $\mathrm{vol} W_Q = \sharp Q$. Since Q is finite, $H_\mathcal{C}^Q$ is an elliptic operator on a bounded domain and $\chi_{(-\infty, \lambda]}(H_\mathcal{C}^Q)$ is trace-class. However, $\chi_{(-\infty, \lambda]}(H_\mathcal{C})$ is not, which is the reason why we need the existence theorem below, and why $H_\mathcal{C}$ may display any interesting spectral features at all.

For $M \in \mathbb{N}$ we denote by $C_M \subset \mathbb{Z}^d$ the cube at the origin with side length $M - 1$, i.e.

$$C_M := \{x \in \mathbb{Z}^d : 0 \le x_j \le M - 1, j = 1, \dots, d\}.$$

Now we are prepared to state our main theorem:

Theorem 2. *Let* $\mathcal{C}\colon \mathbb{Z}^d \to \mathcal{A}$ *be a colouring,* $(U_j)_{j\in\mathbb{N}}$ *a van Hove sequence such that for all patterns* $P \in \bigcup_{M\in\mathbb{N}} \mathcal{P}(C_M)$ *the frequencies* ν_P *exist. Let* $I \subset \mathbb{R}$ *be a finite interval. Then there exists a function* N *belonging to all* $L^p(I)$ *with* $1 \le p < \infty$ *and independent of the van Hove sequence such that for* $j \to \infty$ *we have*

$$(3) \qquad \int_I \left| N(\lambda) - \frac{1}{\sharp U_j} N(\lambda, U_j) \right|^p d\lambda \to 0$$

for any any $p \in [1, \infty)$. *More precisely, for any* $M \in \mathbb{N}$ *the above integral is bounded by*

$$\frac{C}{M} + \left(C(T+C)^{\frac{d}{2}} + c_{p,d} C^{\frac{1}{p}} \right) \frac{\sharp \partial^M U_j}{\sharp U_j} +$$

$$+ C(T+C)^{\frac{d}{2}} \sum_{P \in \mathcal{P}(C_M)} \left| \frac{\sharp P(\mathcal{C}|_{U_j})}{\sharp U_j} - \nu_P \right|$$

where $T = \sup I$. *The dependencies of the constants appearing in the estimate are as follows:* $c_{p,d}$ *depends only on the dimension* d *and the exponent* p, *and* C *depends only on* d *and on (the Kato norm of)* $V_{\mathcal{C},-}$.

Remark 3.
- Obviously, the theorem yields a function $N \in L^p(\mathbb{R})$ such that on each finite interval I the above holds, and we may extend this to (upper) semibounded intervals since our operators are uniformly semibounded below.
- In particular one may choose the van Hove sequence $U_j := C_j$ or a similar family of expanding cubes (if the frequencies exist) so that one gets an explicit $\frac{1}{j}$-decay for the second term in the error estimate. A sequence of cubes is a common choice for defining the IDS.
- In the random setting, introduced in Section 5, there is an alternative definition of the IDS (Pastur-Shubin formula) which coincides with N, as we show there.

3. BOUNDS ON THE SPECTRAL SHIFT FUNCTION OF FACETS

In this section we prove certain bounds on the spectral shift function (SSF) which are needed in Section 4. They concern the SSFs of two electromagnetic Schrödinger operators which differ by an (additional) Dirichlet boundary condition on a facet. This difference can be understood as a generalized (positive) compactly supported potential. Hence we can use results established in [HKN$^+$06], either verbatim or with slight modifications.

Recall that if V_- is in the Kato class, then there exists some number δ smaller than one such that V_- is relatively Δ-bounded with relative bound δ.

We quote Lemma 5 from [HKN$^+$06]:

Lemma 4. *Let* $H^{\mathcal{U}}$ *be a Schrödinger operator defined on the open set* $\mathcal{U} \subset \mathbb{R}^d$ *of finite volume which satisfies Assumption 1. Denote by* E_n *the* n^{th} *eigenvalue counted from below including multiplicity of* $H^{\mathcal{U}}$. *Then there exists a constant* C_1 *such that*

$$(4) \qquad E_n \ge \frac{2\pi(1-\delta)d}{e} \left(\frac{n}{|\mathcal{U}|} \right)^{2/d} - C_1 \qquad \text{for all } n \in \mathbb{N}.$$

The constant C_1 depends on the Kato norm of V_- only. Lemma 4 will be used in the proof of the next result, which is a slight modification of Theorem 1 in [HKN$^+$06].

We will frequently use certain subsets of $d - 1$ dimensional hyperplanes in $\mathbb{R}^d$. They are unit squares in the hyperplanes. More formally we define:

Definition 5. A set $S \subset \mathbb{R}^d$ is called *canonical facet* if there exists a $j \in \{1, \ldots, d\}$ such that

$$S = \{(x_1, \ldots, x_d) \mid x_j = 0 \text{ and } x_i \in [0, 1] \text{ for } i \neq j\}.$$

A set S is called a *facet* if there exists a canonical facet $\tilde{S}$ and a vector $x \in \mathbb{Z}^d$ such that

$$S = x + \tilde{S} := \{y \in \mathbb{R}^d \mid y - x \in \tilde{S}\}.$$

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^d$, S a facet as defined in Definition 5, and $\tilde{\mathcal{U}} := \mathcal{U} \backslash S$. Let H_1 be a Schrödinger operator on $\mathcal{U}$, satisfying Assumption 1. Using quadratic forms we define H_2 as the Dirichlet restriction of H_1 to $\tilde{\mathcal{U}}$. Then the operators e^{-H_1}, e^{-H_2}, and $V_{\text{eff}} := e^{-H_2} - e^{-H_1}$ are well defined by the spectral calculus.

Theorem 6. *(a) The operator V_{eff} is compact.*
(b) Denote by μ_n the nth singular value of V_{eff} counted from above including multiplicity. Then there are finite positive constants c and C_2 such that the singular values of the operator V_{eff} obey

$$(5) \qquad \mu_n \leq C_2 \, e^{-cn^{1/d}}.$$

The constant c may be chosen depending only on d, while C_2 depends only on the Kato-class norm of V_-.

Proof. To prove part (a), i. e. that the operator V_{eff} is compact, it is sufficient to find a family $A_R, R > 0$ of compact operators such that the operator norm of the difference $D_R := V_{\text{eff}} - A_R$ tends to zero as $R \to \infty$. Indeed, such an operator family will be constructed in the proof of the quantitative statement (b). The operators in the family will be even trace-class.

The proof of (b) is almost the same as the one of Theorem 1 in [HKN$^+$06]. We use the same notation as there and explain only the step in the proof which is slightly different in the present setting. Let $B := B_R \subset \mathbb{R}^d$ be an open ball of radius $R > 0$ containing the facet S in its interior, and S^c the complement of S.

Denote by H_1^B the Dirichlet restriction of H_1 to the set $\mathcal{U} \cap B$ and by H_2^B the Dirichlet restriction of H_2 to the set $\tilde{\mathcal{U}} \cap B$. Set

$$D = V_{\text{eff}} - \left(e^{-H_2^B} - e^{-H_1^B}\right).$$

Let $\mathbf{E}_x$ and $\mathbf{P}_x$ denote expectation and probability for Brownian motion, b_t, starting at x. For any open set $U \subset \mathbb{R}^d$ denote by $\tau_U = \inf\{t > 0 \mid b_t \notin U\}$ the first exit time from U. We use the Feynman-Kac-Itô formula to express $e^{-H_1} f$ for $f \in C_c(\mathcal{U})$ as

$$e^{-H_1} f(x) = \mathbf{E}_x \left[e^{-iS_A^1(b)} e^{-\int_0^1 V(b_s) ds} \chi_{\{\tau_{\mathcal{U}} > 1\}}(b) f(b_1) \right]$$

where S_A^t is a real valued stochastic process (Itô integral) corresponding to the magnetic vector potential A of the Schrödinger operator; this representation holds for more general f [Sim79] but a dense set suffices for our purposes. Analogous representations hold for the operators e^{-H_2}, $e^{-H_1^B}$, and $e^{-H_2^B}$ if one replaces the condition $\chi_{\{\tau_{\mathcal{U}}>1\}}$ by

$$\chi_{\{\tau_{\tilde{\mathcal{U}}}>1\}}, \chi_{\{\tau_{\mathcal{U}\cap B}>1\}}, \text{ and } \chi_{\{\tau_{\tilde{\mathcal{U}}\cap B}>1\}}, \text{ respectively.}$$

Since the operator D can be expressed in terms of four different exponentials it follows that

$$(Df)(x) = \mathbf{E}_x\left[\rho(b)\, e^{-iS_A^1(b)} e^{-\int_0^1 V(b_s)ds} f(b_1)\right]$$

where

$$\rho = \chi_{\{\tau_{\mathcal{U}}>1\}} - \chi_{\{\tau_{\tilde{\mathcal{U}}}>1\}} - \chi_{\{\tau_{\mathcal{U}\cap B}>1\}} + \chi_{\{\tau_{\tilde{\mathcal{U}}\cap B}>1\}}.$$

A simple transformation (using $\chi_{\{\tau_{\mathcal{U}}>1\}} - \chi_{\{\tau_{\tilde{\mathcal{U}}}>1\}} = \chi_{\{\tau_{\mathcal{U}}>1\}}(1-\chi_{\{\tau_{\tilde{\mathcal{U}}}>1\}}) = \chi_{\{\tau_{\mathcal{U}}>1\}}\chi_{\{\tau_{\tilde{\mathcal{U}}}\leq1\}} = \chi_{\{\tau_{\mathcal{U}}>1\}}\chi_{\{\tau_{S^c}\leq1\}}$ etc.) shows that

$$\rho = \chi_{\{\tau_{\mathcal{U}}>1\}}\, \chi_{\{\tau_{S^c}\leq1\}}\, \chi_{\{\tau_B\leq1\}}.$$

We abbreviate $\mathcal{B} := \{\tau_{S^c} \leq 1\} \cap \{\tau_B \leq 1\}$. The Hölder inequality implies that

$$|Df|(x) \leq \left(\mathbf{E}_x\left[\chi_{\{\tau_{\mathcal{U}}>1\}}e^{-4\int_0^1 V(b_s)ds}\right]\right)^{1/4}\left(\mathbf{E}_x\left[\chi_{\mathcal{B}}(b)\right]\right)^{1/4}\left(\mathbf{E}_x\left[|f(b_1)|^2\right]\right)^{1/2}.$$

At this point we make the dependence of the operator D on the radius of the ball $B = B_R$ explicit and denote it consequently by D_R. From this point on we can follow exactly the proof of Theorem 1 in [HKN$^+$06] to conclude that

$$\|D_R\| \leq \text{const } \exp\left(\frac{-R^2}{32}\right)$$

With the choice $R = R_n = n^{1/2d}$ the desired estimate (5) follows. $\qquad\square$

The next Lemma is an abstraction of the proof of Theorem 2 in [HKN$^+$06], which in turn relies on [HS02]. The abstract formulation may be of use also in other contexts.

Let A, B be two selfadjoint operators such that $V_{\text{eff}} = e^{-B} - e^{-A}$ is trace class. This implies that the sequence $\mu = \{\mu_n\}_{n\in\mathbb{N}}$ of singular values of V_{eff} (enumerated in decreasing order including multiplicity) converges to zero. Let $F\colon [0,\infty) \to [0,\infty)$ be a convex function with $F(0) = 0$. In particular, F is isotone because for $x \in [0,\infty), \alpha \in [0,1]$ convexity gives $F(\alpha x) = F(\alpha x + (1-\alpha)0) \leq \alpha F(x) + (1-\alpha)F(0) = \alpha F(x)$ which, since $F(x) \geq 0$, implies $F(\alpha x) \leq F(x)$. Set $\phi(n) = F(n) - F(n-1)$ for $n \in \mathbb{N}, n \geq 2$.

Lemma 7. *(a) Let F be as above. Then*

$$(6) \qquad \int_{-\infty}^T F\big(|\xi(\lambda, B, A)|\big)\, d\lambda \leq e^T \langle\phi, \mu\rangle_{\ell^2(\mathbb{N})}$$

(b) Let $h\colon \mathbb{R} \to \mathbb{R}$ be a bounded measurable function with support in $(-\infty, T]$. Then

$$(7) \qquad \int_{\mathbb{R}} h(\lambda)\, \xi(\lambda, B, A)\, d\lambda \leq e^T \langle\phi, \mu\rangle_{\ell^2(\mathbb{N})} + \int_{-\infty}^T G\big(|h(\lambda)|\big)\, d\lambda$$

where G denotes the Legendre transform of F, i.e. $G(y) = \sup\{xy - F(x) \mid x \geq 0\}$ for $y \geq 0$.

Of course the usefulness of the Lemma depends heavily on a priori information about the decay of the sequence μ. However, for the choice of operators $A = H_1$ and $B = H_2$ we do know that the singular values decay at an almost exponential rate.

Remark 8. Depending on how many additional properties we assume for the function F, we obtain correspondingly more information about its Legendre transform G. This will be discussed next.

(1) The following properties hold under no additional assumptions on F: $G(0) = 0$, G is convex (because $xy - F(x)$ is convex in y) and $G(y) \geq 0$ for all y.

(2) If $\lim\limits_{x \to \infty} \frac{F(x)}{x} = \infty$ then G takes on finite values only (and vice versa).

(3) If F is twice differentiable, $f := F' > 0$ and $F'' > 0$ on $[0, \infty)$, then

$$G(y) = \begin{cases} 0 & \text{if } y \leq f(0), \\ yf^{-1}(y) - F(f^{-1}(y)) & \text{if } y > f(0). \end{cases}$$

Here f^{-1} denotes the inverse of the function f. Consequently,

$$\int\limits_{-\infty}^{T} G\big(|h(\lambda)|\big)\, d\lambda = \int\limits_{\{\lambda \leq T : |h(\lambda)| > f(0)\}} \big(|h(\lambda)| f^{-1}(|h(\lambda)|) - F(f^{-1}(|h(\lambda)|))\big)\, d\lambda.$$

(4) If there exist a positive constant C and an exponent $p > 1$ such that $\mu_n \leq Cn^{-p}$, then one can choose the function F as $F(x) = x^{q+1}$, where q is any number smaller than $p - 1$. Indeed, for this choice of F we have $\phi(n) \leq (q+1)n^q$. Thus

$$\sum_n \mu_n \phi(n) \leq (q+1)C \sum_n n^{-p}n^q < \infty.$$

Note that in this case $G(y) = q\left(\frac{y}{q+1}\right)^{\frac{q+1}{q}}$.

(5) If there exist positive constants c, C and p such that $\mu_n \leq Ce^{-cn^p}$ for all $n \in \mathbb{N}$, then for each value of $t < c$, the choice

$$F(x) = \int_0^x \left(e^{ty^p} - 1\right) dy$$

gives a finite right hand side in (6). Indeed, $\phi(n) = \int_{n-1}^n \left(e^{ty^p} - 1\right) dy \leq e^{tn^p}$ and thus

$$\langle \phi, \mu \rangle_{\ell^2(\mathbb{N})} \leq C \sum_n e^{-cn^p} e^{tn^p} < \infty.$$

The Legendre transform for such a choice of F satisfies

$$G(y) \leq yf^{-1}(y) = \left(\frac{\log(1+y)}{t}\right)^{1/p} \quad \text{for all } y \geq 0.$$

Thus in this specific case inequality (7) reads

$$(8) \quad \int_{\mathbb{R}} h(\lambda) \, \xi(\lambda, B, A) \, d\lambda \leq e^T C \sum_{n \in \mathbb{N}} e^{-(c-t)n^p} +$$

$$\int |h(\lambda)| \left(\frac{\log(1 + |h(\lambda)|)}{t} \right)^{1/p} d\lambda$$

which recovers the result of [HKN+06].

Now we prove Lemma 7.

Proof. Since V_{eff} is trace class but not necessarily the operator difference $A - B$, the SSF is defined via the invariance principle

$$\int_{-\infty}^{T} F(|\xi(\lambda, B, A)|) \, d\lambda = \int_{-\infty}^{T} F(|\xi(e^{-\lambda}, e^{-B}, e^{-A})|) \, d\lambda.$$

Now a change of variables gives us

$$\int_{-\infty}^{T} F(|\xi(e^{-\lambda}, e^{-B}, e^{-A})|) \, d\lambda \leq e^T \int_{e^{-T}}^{\infty} F(|\xi(s, e^{-B}, e^{-A})|) \, ds.$$

To the last expression we can apply the estimate of [HS02] and bound it above by

$$\int_{e^{-T}}^{\infty} F(|\xi(s, e^{-B}, e^{-A})|) \, ds \leq \sum_{n \in \mathbb{N}} \mu_n(V_{\mathrm{eff}}) \phi(n).$$

This establishes claim (a). To prove (b), we note that by the very definition of the Legendre transform, the Young inequality

$$|h \cdot \xi| \leq F(|\xi|) + G(|h|).$$

holds. Integrating over λ we obtain

$$\int h(\lambda)\xi(\lambda) \, d\lambda \leq \int_{-\infty}^{T} F(|\xi(\lambda, B, A)|) \, d\lambda + \int G(|h(\lambda)|) \, d\lambda.$$

Together with (a) this completes the proof. $\qquad \square$

4. ALMOST ADDITIVITY FOR THE EIGENVALUE COUNTING FUNCTIONS

In this section we prove Theorem 2. For this aim we will apply a Banach space-valued ergodic theorem obtained in [LMV08]. Actually, for our purposes it will be convenient to quote a slightly streamlined version of this result from [LSV10]. To spell it out we need to introduce the notion of a *boundary term* and the properties of *almost additivity* and *invariance*.

Definition 9. A function $b \colon \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d) \to [0, \infty)$ is called a *boundary term* if the following three properties hold:

(i) $b(Q) = b(Q + x)$ for all $x \in \mathbb{Z}^d$ and all $Q \in \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d)$,

(ii) $\lim_{j \to \infty} \frac{b(U_j)}{\sharp U_j} = 0$ for any van Hove sequence $(U_j)_{j \in \mathbb{N}}$ and

(iii) there exists a constant $D \in (0, \infty)$ such that

$$b(Q) \leq D \, \sharp Q \quad \text{for all} \quad Q \in \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d)$$

Definition 10. Let $(X, \|\cdot\|)$ be a Banach space and F a function $\mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d) \to X$.

(a) The function F is said to be *almost-additive* if there exists a boundary term b such that

$$\|F(\cup_{k=1}^m Q_k) - \sum_{k=1}^m F(Q_k)\| \leq \sum_{k=1}^m b(Q_k)$$

for all $m \in \mathbb{N}$ and all pairwise disjoint sets $Q_k \in \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d)$, $k = 1, \ldots, m$.

(b) Let $\mathcal{C} : \mathbb{Z}^d \longrightarrow \mathcal{A}$ be a colouring. The function F is said to be *$\mathcal{C}$-invariant* if

$$F(Q) = F(Q + x)$$

whenever $x \in \mathbb{Z}^d$ and $Q \in \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d)$ obey $\mathcal{C}|_Q + x = \mathcal{C}|_{Q+x}$.

In this case there exists a function $\widetilde{F}$ defined on the (classes of) patterns such that $\widetilde{F}(P) = F(Q)$ if $\mathcal{C}|_Q = P$.

If $F \colon \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d) \to X$ is almost additive and invariant there exists a $K \in (0, \infty)$ such that

$$(9) \qquad \|F(Q)\| \leq K \, \sharp Q \qquad \text{for all } Q \in \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d).$$

Now we are in the position to quote the Banach space valued ergodic theorem from [LMV08], see [LSV10, §5.1] as well.

Theorem 11. *Let $\mathcal{A}$ be a finite set of colours, $\mathcal{C} \colon \mathbb{Z}^d \to \mathcal{A}$ a colouring and (U_j) a van Hove sequence along which the frequencies of all patterns $P \in \bigcup_{M \in \mathbb{N}} \mathcal{P}(C_M)$ exist. Let $F : \mathcal{F}(\mathbb{Z}^d) \to X$ be a $\mathcal{C}$-invariant and almost-additive function. Then the limit*

$$\overline{F} := \lim_{j \to \infty} \frac{F(U_j)}{\sharp U_j} = \lim_{M \to \infty} \sum_{P \in \mathcal{P}(C_M)} \nu_P \frac{\widetilde{F}(P)}{\sharp C_M}.$$

exists in X. Furthermore, for $j, M \in \mathbb{N}$ the bound
$$(10)$$
$$\left\| \overline{F} - \frac{F(U_j)}{\sharp U_j} \right\| \leq 2 \frac{b(C_M)}{M^d} + (K + D) \frac{\sharp \partial^M U_j}{\sharp U_j} + K \sum_{P \in \mathcal{P}(C_M)} \left| \frac{\sharp_P(\mathcal{C}|_{U_j})}{\sharp U_j} - \nu_P \right|.$$

holds.

We want to apply the ergodic theorem to the eigenvalue counting functions of Schrödinger operators, considered as elements of $X := L^p(I)$ for a fixed finite interval $I \subset \mathbb{R}$ and $p \in [1, \infty)$.

More precisely, we study the function

$$F : \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d) \to L^p(I), \quad Q \mapsto N(\cdot, H^Q)$$

with $N(\lambda, H^Q) := \mathrm{Tr}\chi_{(-\infty, \lambda]}(H^Q)$ for $\lambda \in \mathbb{R}$. Note that this notation is slightly different from the one used in Theorem 2. The reason is that in the proofs we use a more general class of operators than which was necessary to formulate the main result and thus need a bit more flexibility. To conclude Theorem 2 we need to show that F fulfils the hypotheses of Theorem 11. This is done in the following

Lemma 12. *The function $F : \mathcal{F}_{fin}(\mathbb{Z}^d) \to L^p(I)$ is invariant and almost-additive.*

Proof. Note that H is a local operator, thus $H^{\mathcal{U}}$ depends only on $A|_{\mathcal{U}}$ and $V|_{\mathcal{U}}$, for any open $\mathcal{U} \subset \mathbb{R}^d$. The translation operator $T_y f(x) = f(x - y)$ is unitary for any $y \in \mathbb{R}^d$. Thus the spectrum of H, resp. $H^{\mathcal{U}}$, is invariant under conjugation by T_y. It follows that the function F is $\mathcal{C}$-invariant.

To prove almost-additivity, let $Q = \bigcup_{k=1}^m Q_k$ with pairwise disjoint $Q_k \in \mathcal{F}_{\text{fin}}(\mathbb{Z}^d), k = 1, \ldots, m$. We need to compare $F(Q)$ to $\sum_{k=1}^m F(Q_k)$. Note that $W_Q = \bigcup_{k=1}^m W_{Q_k}$ but the W_{Q_k} need not be pairwise disjoint because their boundaries can touch. There are two extreme cases:

(1) All W_{Q_k} are pairwise disjoint. Then $\mathring{W}_Q = \bigcup_{k=1}^m \mathring{W}_{Q_k}$, and consequently $H^Q = \bigoplus_{k=1}^m H^{Q_k}$ and $F(Q) = \sum_{k=1}^m F(Q_k)$.
(2) No W_{Q_k} is disjoint from all others. Since ∂W_{Q_k} consists of at most $2d \sharp \partial Q_k$ facets (where ∂Q_k denotes the combinatorial boundary of $Q_k \subset \mathbb{Z}^d$) the sets $\mathring{W}_Q$ and $\bigcup_{k=1}^m \mathring{W}_{Q_k}$ differ by at most $2d \sum_{k=1}^m \sharp \partial Q_k$ facets.

In fact, the latter case gives an upper bound for the general case (where the "isolated" Q_k simply do not contribute), and we can drop the factor 2 because touching facets need to be counted once only (they are counted twice in the sum).

So, let $M \leq d \sum_{k=1}^m \sharp \partial Q_k$ be the number of facets by which $\mathring{W}_Q$ and $\bigcup_{k=1}^m \mathring{W}_{Q_k}$ differ and enumerate them arbitrarily as $S_1, \ldots, S_M$. Set $H_0 := H^Q$ and

$$H_j := H^{U_j} \text{ where } U_j := W_Q \setminus \bigcup_{i=1}^j S_i$$

for $j = 1, \ldots, M$. Clearly, $H_M = \bigoplus_{k=1}^m H^{Q_k}$ and $N(\cdot, H_M) = \sum_{k=1}^m F(Q_k)$. This relation allows us to write the difference that we want to estimate as a sum of SSFs:

$$F(Q) - \sum_{k=1}^m F(Q_k) = \sum_{j=1}^M \left(N(\cdot, H_{j-1}) - N(\cdot, H_j) \right)$$
$$= \sum_{j=1}^M \xi(\cdot, H_{j-1}, H_j)$$

H_{j-1} and H_j differ exactly by a Dirichlet condition at one facet S_j so that Theorem 6 applies and gives the estimate $\mu_n \leq Ce^{-cn^{1/d}}$ for the singular values. Now we can apply Lemma 7 with $T := \sup I$, $A := H_j$, $B := H_{j-1}$ and $F(x) := x^p$. Then inequality (6) reads

$$(11) \qquad \int_I |\xi(\cdot, H_{j-1}, H_j)|^p \leq e^T \sum_n C_2 p n^{p-1} e^{-cn^{1/d}} =: \tilde{C}^p$$

with a constant $\tilde{C}$ which is independent of j. The triangle inequality thus gives

$$\left\| F(Q) - \sum_{k=1}^{m} F(Q_k) \right\|_{L^p(I)} \leq M\tilde{C} \leq d\,\tilde{C} \sum_{k=1}^{m} \sharp\partial Q_k$$

$$=: \sum_{k=1}^{m} b(Q_k).$$

The function $b : \mathcal{F}_{\mathrm{fin}}(\mathbb{Z}^d) \to \mathbb{R}, Q \mapsto d\,\tilde{C}\,\sharp\partial Q$ satisfies the three conditions required for a boundary term. $\qquad\square$

Proof of Theorem 2. Given the previous Lemma, we can apply the abstract ergodic theorem to our counting functions associated to finite subsets Q. In order to check the form of the error estimate, we note that the constant $\tilde{C}$ in equation (11) satisfies

$$\tilde{C}^p = C_2\, e^T \operatorname{const}(p,d)$$

with C_2 being the constant from Theorem 6 depending on the Kato-norm of $V_{\mathcal{C},-}$. Also,

$$b(Q) = d\tilde{C}\,\sharp\partial Q$$

so that the bound D on b in the general abstract ergodic theorem can be taken to be $d\tilde{C}$. It follows that $\frac{2b(\tilde{C}_M)}{M^d} \leq 4d^2\tilde{C}/M$.

Finally, the uniform lower estimate on the eigenvalues established in Lemma 4 gives a uniform upper estimate on the number of eigenvalues in $(-\infty, T]$, namely

$$(T + C_1)^{d/2} \left(\frac{e}{2\pi(1-\delta)}d \right)^{d/2} |\mathcal{U}|.$$

Hence we can take

$$K = C_3 \left(T + C_1\right)^{\frac{d}{2}}$$

as the uniform bound on the function F in the abstract ergodic theorem. Here the constants C_1, C_2, C_3 depend only on d and the Kato norm of $V_{\mathcal{C},-}$. $\qquad\square$

5. Application to random operators

In order to apply our results to random operators we quote the necessary random versions of the definition and main theorem from [GLV08]: Let $(\Omega, \mathbb{P})$ be a probability space such that $\mathbb{Z}^d$ acts ergodically on $(\Omega, \mathbb{P})$. We denote the $\mathbb{Z}^d$-action on Ω by $x\colon \omega \mapsto \omega - x$. A random $\mathcal{A}$-colouring is a map

$$\mathcal{C} : \Omega \longrightarrow \bigotimes_{\mathbb{Z}^d} \mathcal{A} \quad \text{with} \quad \mathcal{C}(\omega - y)_{x-y} = \mathcal{C}(\omega)_x$$

for all $x, y \in \mathbb{Z}^d$. Note that for each fixed ω we obtain a (usual) $\mathcal{A}$-colouring. By the (usual) ergodic theorem for scalar functions the frequencies of patterns exist almost surely. Thus we can apply our abstract Banach space valued ergodic theorem:

Theorem 13. *Let $\mathcal{A}$ be a finite set, $\mathcal{C}$ be a random $\mathcal{A}$-colouring and $(X, \|\cdot\|)$ a Banach space. Let $(U_j)_{j\in\mathbb{N}}$ be a van Hove sequence. For each fixed $\omega \in \Omega$ let $F_\omega : \mathcal{F}_{fin}(\mathbb{Z}^d) \longrightarrow X$ be a $\mathcal{C}(\omega)$-invariant, almost-additive bounded function. Assume that the family $(F_\omega)_{\omega\in\Omega}$ is $\mathbb{Z}^d$-homogeneous, i.e. $F_{\omega+x}(Q+x) = F_\omega(Q)$ for all $x \in \mathbb{Z}^d, Q \in \mathcal{F}_{fin}(\mathbb{Z}^d)$. Then, for almost every $\omega \in \Omega$ the limits*

$$\overline{F}_\omega := \lim_{j\to\infty} \frac{F_\omega(U_j)}{|U_j|} = \lim_{M\to\infty} \sum_{P\in\mathcal{P}(C_M)} \nu_P \frac{\widetilde{F}(P)}{|C_M|}$$

exist in the topology of $(X, \|\cdot\|)$ and are equal. In particular, $\overline{F}_\omega$ is almost surely independent of ω.

Concretely, we apply this to colourings given by local models for the potentials like in Section 5, but now with a randomly chosen colouring so that all operators and counting functions additionally depend on the random variable ω. For the formulation of the theorem we introduce a distribution function $N\colon \mathbb{R} \to \mathbb{R}$ defined by a trace per unit volume formula (sometimes called Pastur-Shubin formula)

$$(12) \qquad N(\lambda) := \int_\Omega \mathrm{Tr}\left[\chi_{W_0}\chi_{(-\infty,\lambda]}(H_\omega)\right] d\mathbb{P}(\omega).$$

By applying the random version of the ergodic theorem we get the random version of Theorem 2:

Theorem 14. *Let $\mathcal{C} : \Omega \longrightarrow \bigotimes_{\mathbb{Z}^d} \mathcal{A}$ be a random colouring, $(U_j)_{j\in\mathbb{N}}$ a van Hove sequence. Let $I \subset \mathbb{R}$ be a finite interval and $p \in [1,\infty)$. Then for $j \to \infty$ we have for almost all $\omega \in \Omega$*

$$(13) \qquad \int_I \left| N(\lambda) - \frac{1}{\sharp U_j} N_\omega(\lambda, U_j) \right|^p d\lambda \to 0.$$

Remark 15. Of course there are similar error estimates as in Theorem 2.

Proof. The existence of the limit in Equation (13) follows directly from our abstract theorem since we checked all requirements in the proof of Theorem 2 already.

For the proof of the Shubin-Pastur formula (12) we use a variation of the proof of Theorem 3 in [GLV07]: First notice that

$$N(\lambda) = \frac{1}{\sharp U_j} \int_\Omega \mathrm{Tr}\left[\chi_{W^{U_j}}\chi_{(-\infty,\lambda]}(H_\omega)\right] d\mathbb{P}(\omega)$$

independently of U_j due to additivity and invariance. Now it suffices to show that

$$(*) \qquad \frac{1}{\sharp U_j}\mathrm{Tr}\left[\chi_{W^{U_j}}e^{-tH_\omega} - e^{-tH_\omega^{U_j}}\right] \to 0, \quad j \to \infty$$

for all t. We use the abbreviation $\oplus H_\omega = H_\omega^{U_j} \oplus H_\omega^{\mathbb{Z}^d \setminus U_j}$ and the linearity of the trace to conclude

$$\mathrm{Tr}\left[\chi_{W^{U_j}}e^{-tH_\omega} - e^{-tH_\omega^{U_j}}\right]$$

$$= \mathrm{Tr}\left[\chi_{W^{U_j}}(e^{-tH_\omega} - e^{-t\oplus H_\omega})\right] + \mathrm{Tr}\left[\chi_{W^{U_j}}e^{-t\oplus H_\omega} - e^{-tH_\omega^{U_j}}\right].$$

The second term actually vanishes. Thus it suffices to estimate the first term, which we do next. Let us note that for a compact operator K and a bounded operator B on the same Hilbert space the corresponding singular values obey the relation

$$\mu_n(BK) \leq \|B\|\mu_n(K).$$

In particular this holds for $K = V_{\text{eff}}$ as in Theorem 6 and $B = \chi_{W^{U_j}}$. Now we can proceed exactly as in Lemma 12, using that the addition of the boundary conditions by which $\oplus H_\omega$ differs from H_ω gives a boundary term in $L^p(I)$ for the corresponding SSF. Now the claim (*) follows from the van Hove property. $\square$

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