Optimal Sobolev embeddings for symmetric gradients

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Main question

For a functions space X find smallest function space Y

s.t. for all $\mathbf{u} \in C_c^\infty(\mathbb{R}^n)$ (for all $\mathbf{u} \in E_0^1(\mathbb{R}^n)$)

$$\|\mathbf{u}\|_{Y(\mathbb{R}^n)} \leq c\,\|\varepsilon(\mathbf{u})\|_{X(\mathbb{R}^n)}$$

- $\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$ for $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$;
- First question: X and Y are r.i. spaces (Lebesgue, Lorentz and Orlicz spaces);
- Second question: X is r.i. space and Y is a space of continuous functions (with given modulus of continuity).

L^p-spaces are not rich enough

- If $X = L^p$ for $1 < p < n$ we have Korn's inequality, so Y is the space from the known Sobolev embedding;
- If $X = L^n$ there is no optimal (smallest possible) Lebesgue space Y s.t.

$$\|u\|_Y \leq c\,\|\nabla u\|_{L^n}.$$

  Can be answered in the class of Orlicz spaces with $Y = \exp(L^{n'})$.
- Iteration of optimal embeddings fails: let $n = 2$

$$W^{2,1} \hookrightarrow W^{1,2} \hookrightarrow \cap_p L^p, \quad W^{2,1} \hookrightarrow L^\infty;$$
$$W^{2,1} \hookrightarrow W^1 L^{2,1} \hookrightarrow L^\infty.$$

Motivation 1: non-Newtonian fluids

Eyring suggested in 1936 the constitutive law

$$\mathbf{S}(\varepsilon(\mathbf{v})) = \nu_0 \frac{\operatorname{arsinh}(\lambda|\varepsilon(\mathbf{v})|)}{\lambda|\varepsilon(\mathbf{v})|} \approx \frac{\log(1+|\varepsilon(\mathbf{v})|)}{|\varepsilon(\mathbf{v})|}\varepsilon(\mathbf{v}).$$

- The natural function space (for $\varepsilon(\mathbf{u})$) is the Orlicz space generated by $A(t) = t\ln(1+t)$; Korn's inequality fails (Breit-Diening, JMFM, '12).
- Existence of weak solutions in 2D is proved by Breit-Diening-Fuchs (JDE, '12). For $\operatorname{div}(\mathbf{u}\otimes\mathbf{u})$ one needs compactness onto L^2.

Motivation 1: non-Newtonian fluids

Theorem (Cianchi, JFA, 2015, Breit-Cianchi-Diening, SIMA, 2017)

The inequality

$$\|\nabla \mathbf{u}\|_{L^B} \leq K \|\varepsilon(\mathbf{u})\|_{L^A}$$

for all $\mathbf{u} \in C_c^\infty(\mathbb{R}^n)$ with two Young functions A and B holds iff A and B satisfy ....

- Integrabilities are optimal (in the sense of Orlicz spaces);
- $A = B$ if A satisfies Δ_2 and ∇_2 (e.g. $A(t) = t^p$);
- $A(s) = s\ln(1+s)$, $B(s) = s$;
- $A(s) = \infty\chi_{(1,\infty)}$, $B(s) = s(e^s - 1)$.

Motivation 2: Orlicz-Sobolev embeddings

Optimal embedding for Young function A (Cianchi, 1996, IUMJ)

$$\|u\|_{L^{A_n}} \leq c\,\|\nabla u\|_{L^A}, \quad A_n(s) = \int_0^s r^{n'-1}(\Phi_n^{-1}(r^n))^{n'},$$

$$\Phi_n(s) = \int_0^s \frac{\tilde{A}(t)}{t^{1+n'}}, \quad \tilde{A}(t) = \sup_{s\geq 0}[st - A(s)].$$

- Recovers standard cases such as $W^{1,p} \hookrightarrow L^{pn/(n-p)}$ and $W^{1,n} \hookrightarrow \exp(L^{n'})$;
- Can we produce the same embedding if ∇u is replaced by $\varepsilon(\mathbf{u})$?

Rearrangement invariant function spaces

Let $\|\cdot\|_{X(0,|\Omega|)}$ be a rearrangement-invariant function norm

$$\|\mathbf{u}\|_{X(\Omega)} = \|\mathbf{u}^*\|_{X(0,|\Omega|)}$$

- *Decreasing rearrangement* of a measurable function $\mathbf{u}$:

$$\mathbf{u}^*(s) = \inf\{t \geq 0 : |\{x \in \Omega : |\mathbf{u}(x)| > t\}| \leq s\} \quad \text{for } s \in [0,\infty);$$

- $X(\Omega)$ includes all functions s.t. $\|\mathbf{u}\|_{X(\Omega)} < \infty$;
- We have $\|\mathbf{u}\|_{L^p(\Omega)} = \|\mathbf{u}^*\|_{L^p(0,|\Omega|)}$ for all $1 \leq p \leq \infty$;
- W^1X: u and ∇u in X; E^1X: $\mathbf{u}$ and $\varepsilon(\mathbf{u})$ in X.

Optimal embeddings in r.i. spaces (1)

Theorem (Breit-Cianchi): Let X and Y be r.i. spaces

$$E_0^1 X(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^n) \quad \Leftrightarrow \quad W_0^1 X(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^n).$$

- Standard and symmetric Sobolev embeddings are equivalent!
- Holds iff for any non-increasing function $f : (0, |\Omega|) \to [0, \infty)$

$$\left\| \int_s^\infty f(r) r^{-1+\frac{1}{n}}\, dr \right\|_{Y(0,|\Omega|)} \leq c_3 \|f\|_{X(0,\infty)};$$

- Version for (ε, δ)-domains with $\mathcal{L}^n(\Omega) < \infty$;
- Version for W_0^1 and E_0^1 for domains with $\mathcal{L}^n(\Omega) < \infty$;
- Special case: Orlicz-Sobolev embedding as in Cianchi '96, e.g.

$$E^1 L^n (log L)^{n-1} \hookrightarrow \exp\exp L^{n'}.$$

Optimal embeddings in r.i. spaces (2)

Main idea: interpolate endpoint embeddings

- Spector-Van Schaftingen (2019)

$$E^1L^1(\mathbb{R}^n) \hookrightarrow L^{n',1}(\mathbb{R}^n);$$

- By Korn's inequality

$$E^1L^{n,1}(\mathbb{R}^n) \hookrightarrow W^1L^{n,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n);$$

- The key: representation formula for the K-functional

$$K(\mathbf{u}, t, E_0^1L^1(\mathbb{R}^n), E_0^1L^\infty(\mathbb{R}^n)) \approx \int_0^t \varepsilon(\mathbf{u})^*(s).$$

Optimal embeddings in C^σ (1)

For an r.i. space X with $\lim_{r\to 0}\sigma_X(r)=0$

$$E^1X(\Omega)\hookrightarrow C^{\sigma_X}(\Omega)\quad (\varepsilon,\delta)\text{-domain},$$
$$E^1_0X(\Omega)\hookrightarrow C^{\sigma_X}(\Omega)\quad \text{bounded domain},$$
$$\sigma_X(s)=\|r^{-\frac{1}{n'}}\chi_{(0,s^n)}\|_{X_e{}'}+s\,\|r^{-1}\chi_{(s^n,R)}\|_{X_e{}'}.$$

- $C^\sigma(\Omega)$ space of all functions with

$$\|\mathbf{u}\|_{C^\sigma(\Omega)}=\|\mathbf{u}\|_{C^0(\Omega)}+\sup_{x,y\in\Omega}\frac{|\mathbf{u}(x)-\mathbf{u}(y)|}{\sigma(|x-y|)}<\infty;$$

- Examples: $\sigma_{L^\infty}\approx r\log(1/r)$ and $\sigma_{\exp(L^\beta)}\approx r(\log(1/r))^{1+1/\beta}$.

Optimal embeddings in C^σ (2)

Ideas of the proof:

- Sufficiency: use singular integral representation

$$\mathbf{u}(x) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \mathcal{A}\varepsilon(\mathbf{u})(y) \frac{x-y}{|x-y|^n} \,\mathrm{d}y$$

- Necessity: consider the function $\mathbf{u} : \Omega \to \mathbb{R}^n$ given by

$$\mathbf{u}(x) = \begin{cases} \mathbf{Q}\, x \int_{\omega_n |x|^n}^1 f(r) r^{-1} \,\mathrm{d}r & \text{if } x \in B \\ 0 & \text{if } x \in \Omega \setminus B \end{cases}$$

$$\Rightarrow c \geq \sup_{\mathbf{u} \in E_0^1 X(\Omega)} \frac{|\mathbf{u}(x_1, 0 \ldots, 0) - \mathbf{u}(0, 0, \ldots, 0)|}{\sigma(|x_1|) \|\varepsilon(\mathbf{u})\|_{X(\Omega)}}$$

$$\geq c'' \frac{|x_1| \, \|r^{-1} \chi_{(\omega_n |x_1|^n, 1)}(r)\|_{X'(0,\infty)}}{\sigma(|x_1|)} \geq c''' \frac{\varrho_X(|x_1|)}{\sigma(|x_1|)}.$$

Classical Sobolev spaces

DeVoire-Scherer (Ann. Math., '79)

$$K(u,t,W^{1,1}(\mathbb{R}^n),W^{1,\infty}(\mathbb{R}^n)) \approx \int_0^t u^*(s) + \int_0^t (\nabla u)^*(s).$$

- Inequality "$\geq$" trivial due to

$$K(u,t,L^1(\mathbb{R}^n),L^\infty(\mathbb{R}^n)) \approx \int_0^t u^*(s);$$

- Approach via maximal function and Lipschitz truncation by Calderón-Milman '83.

Symmetric Lipschitz truncation

Given $\mathbf{u} \in E^1 L^1(\mathbb{R}^n)$ and $\theta, \lambda > 0$

$$\mathcal{O}_{\theta,\lambda} = \{x \in \mathbb{R}^n : \ M(\mathbf{u}) > \theta\} \cup \{x \in \mathbb{R}^n : \ M(\varepsilon(\mathbf{u})) > \lambda\}.$$

Whitney covering of $\mathcal{O}_{\theta,\lambda}$ and local projections onto rigid motions.
Truncate $\mathbf{u}$ to $\mathbf{u}_{\theta,\lambda}$ (following Breit-Diening-Fuchs, JDE, '12) s.t.

- $\mathbf{u} = \mathbf{u}_{\theta,\lambda}$ outside $\mathcal{O}_{\theta,\lambda}$;
- $|\mathbf{u}_{\theta,\lambda}| \lesssim \theta$ and $|\varepsilon(\mathbf{u}_{\theta,\lambda})| \lesssim \lambda$;
- $\mathbf{u} \in E^1_0 L^1(\Omega)$ implies $\mathbf{u}_{\theta,\lambda} \in E^1_0 L^\infty(\Omega)$.

Proof (1)

Let $\mathbf{u} \in E^1L^1(\mathbb{R}^n)$, fix $t > 0$, set $\theta = M(\mathbf{u})^*(t)$, $\lambda = M(\varepsilon(\mathbf{u}))^*(t)$.

Set $\mathbf{u}_1 = \mathbf{u}_{\theta,\lambda}$ and $\mathbf{u}_0 = \mathbf{u} - \mathbf{u}_1$, thus

$$K(\mathbf{u}, t, E^1L^1(\mathbb{R}^n), E^1L^\infty(\mathbb{R}^n)) \leq \|\mathbf{u}_0\|_{E^1L^1(\mathbb{R}^n)} + t\|\mathbf{u}_1\|_{E^1L^\infty(\mathbb{R}^n)}.$$

$$\begin{aligned}
\|\mathbf{u}_1\|_{E^1L^\infty} &= \|\mathbf{u}_1\|_{L^\infty} + \|\varepsilon(\mathbf{u}_1)\|_{L^\infty} \lesssim \theta + \lambda \\
&= M(\mathbf{u})^*(t) + M(\varepsilon(\mathbf{u}))^*(t) \\
&\lesssim \frac{1}{t}\left(\int_0^t \mathbf{u}^*(s) + \int_0^t \varepsilon(\mathbf{u})^*(s)\right), \\
\|\mathbf{u}_0\|_{E^1L^1} &\leq \|\mathbf{u}\chi_{\mathcal{O}_{\theta,\lambda}}\|_{L^1} + \|\varepsilon(\mathbf{u})\chi_{\mathcal{O}_{\theta,\lambda}}\|_{L^1} \\
&\quad + \|\mathbf{u}_1\chi_{\mathcal{O}_{\theta,\lambda}}\|_{L^1} + \|\varepsilon(\mathbf{u}_1)\chi_{\mathcal{O}_{\theta,\lambda}}\|_{L^1},
\end{aligned}$$

Proof (2)

Using $|\mathcal{O}_{\theta,\lambda}| \le ct$ we have

$$\|\mathbf{u}\chi_{\mathcal{O}_{\theta,\lambda}}\|_{L^1} \le \|\mathbf{u}^*\chi_{(0,ct)}\|_{L^1(0,\infty)} \lesssim \|\mathbf{u}^*\chi_{(0,t)}\|_{L^1(0,\infty)} = \int_0^t \mathbf{u}^*(s)$$

$$\|\varepsilon(\mathbf{u})\chi_{\mathcal{O}_{\theta,\lambda}}\|_{L^1} \lesssim \int_0^t \varepsilon(\mathbf{u})^*(s),$$

$$\begin{aligned}\|\mathbf{u}_1\chi_{\mathcal{O}_{\theta,\lambda}}\|_{L^1} + \|\varepsilon(\mathbf{u}_1)\chi_{\mathcal{O}_{\theta,\lambda}}\|_{L^1} &\lesssim t\,\theta + t\lambda \\ &= t\big(M(\mathbf{u})^*(t) + M(\varepsilon(\mathbf{u}))^*(t)\big), \\ &\lesssim \left(\int_0^t \mathbf{u}^*(s) + \int_0^t \varepsilon(\mathbf{u})^*(s)\right).\end{aligned}$$

Proof (3)

- For $\mathbf{u} \in E^1L^1(\mathbb{R}^n) + E^1L^\infty(\mathbb{R}^n)$ use Lipschitz truncation;
- For the K-functional on bounded domains Ω use extension operator with the property $\varepsilon(\mathscr{E}_\Omega \mathbf{u}) = \mathscr{L}^1\varepsilon(\mathbf{u}) + \mathscr{L}^2\mathbf{u}$;
- For $\mathbf{u} \in E^1_0L^1(\mathbb{R}^n) + E^1_0L^{n,1}(\mathbb{R}^n) = E^1_0L^1(\mathbb{R}^n)$ we have

$$K(E^1_0L^1(\mathbb{R}^n), E^1_0L^{n,1}(\mathbb{R}^n)) \approx \int_0^{t^{n'}} \varepsilon(\mathbf{u})^*(s) + t\int_{t^{n'}}^\infty \varepsilon(\mathbf{u})^*(s)s^{-\frac{1}{n'}},$$

$$K(t, \mathbf{u}; L^{n',1}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) \approx \int_0^{t^{n'}} s^{-\frac{1}{n}}\mathbf{u}^*(s);$$

- The Sobolev embedding follows then from

$$\int_0^t s^{-\frac{1}{n}}\mathbf{u}^*(s) \lesssim \int_0^t s^{-\frac{1}{n}} \int_s^\infty \varepsilon(\mathbf{u})^*(r)r^{-\frac{1}{n'}}.$$

Extension operator (1)

Assume that Ω is an (ε, δ)-domain in $\mathbb{R}^n$.

There is extension operator $\mathscr{E}_\Omega : L^1_{\mathrm{loc}}(\Omega) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$:

$$\mathscr{E}_\Omega : E^1L^1(\Omega) \to E^1L^1(\mathbb{R}^n), \ \mathscr{E}_\Omega : E^1L^\infty(\Omega) \to E^1L^\infty(\mathbb{R}^n).$$

- We have $\boldsymbol{\varepsilon}(\mathscr{E}_\Omega \mathbf{u}) = \mathscr{L}^1 \boldsymbol{\varepsilon}(\mathbf{u}) + \mathscr{L}^2 \mathbf{u}$;
- Using the K-functional: $\mathscr{E}_\Omega : E^1X(\Omega) \to E^1X(\mathbb{R}^n)$;
- Construction by Whitney covering and reflection (following Jones, Acta. Math. '81);
- Improves extension operator by Gmeineder-Raita (JFA, '19).

Extension operator (2)

Assume that Ω is an (ε, δ)-domain in $\mathbb{R}^n$

$$\mathscr{E}_\Omega \mathbf{u} = \begin{cases} \mathbf{u} & \text{in } \Omega \\ \sum_{j \in \mathbb{M}} \varphi_j \widetilde{\mathbf{u}}_j & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

- $\widetilde{\mathbf{u}}_j(x) = \widetilde{\mathcal{R}}_j \mathbf{u}(\theta(x - x_j) + \widetilde{x}_j)$ for $j \in \mathbb{M}$, where for $\mathbf{u} \in E^1 L^p(\Omega)$

$$\mathbf{u}(x) = \mathcal{R}_\Omega(\mathbf{u})(x) + \mathscr{L}_\Omega(\varepsilon(\mathbf{u}))(x) \quad \text{for a.e. } x \in \Omega;$$

- $\{Q_j\}$ is covering of $\mathbb{R}^n \setminus \overline{\Omega}$ and $\{\widetilde{Q}_j\}$ covering of Ω;
- $\{Q_j\}_{j \in \mathbb{M}}$ cubes close to $\partial\Omega \rightsquigarrow$ associated cubes from $\{\widetilde{Q}_j\}$.