Long-time behaviour of stochastically forced compressible fluid flows

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Navier–Stokes equations

Find velocity $\mathbf{u} : Q \to \mathbb{R}^d$ and density $\varrho : Q \to \mathbb{R}$ satisfying the

system of partial differential equations

$$\begin{cases} \partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) = \operatorname{div}\mathbf{S} - a\nabla\varrho^\gamma & \text{in } Q, \\ \partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 & \text{in } Q, \\ \mathbf{u} = 0 & \text{on } \partial\mathcal{O}, \\ \varrho(0,\cdot) = \varrho_0 & \text{in } \mathcal{O}, \\ \varrho(0,\cdot)\mathbf{u}(0,\cdot) = \mathbf{q}_0 & \text{in } \mathcal{O}, \end{cases}$$

- $Q := (0,T) \times \mathcal{O}$ with $\mathcal{O} \subset \mathbb{R}^d$, $d \in \{2,3\}$, and $T > 0$;
- $\mathbf{S} : Q \to \mathbb{R}^{d\times d}$ is given by Newton's law

$$\mathbf{S} = \mu\Big(\frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2} - \frac{1}{3}\operatorname{div}\mathbf{u}\, I\Big) + \big(\eta + \tfrac{2}{3}\big)\operatorname{div}\mathbf{u}\, I.$$

Momentum equation

Weak formulation of the momentum equation:

find $\mathbf{u}, \varrho$ such that for all $\varphi \in C_c^\infty(Q)$

$$\int_Q \varrho \mathbf{u} \cdot \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t = \mu \int_Q \nabla \mathbf{u} : \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t + (\eta + \mu) \int_Q \operatorname{div} \mathbf{u} \operatorname{div} \varphi$$

$$- \int_Q (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_Q a\varrho^\gamma \operatorname{div} \varphi \,\mathrm{d}x\,\mathrm{d}t$$

- Function space for weak solutions

$$\mathbf{u} \in L^2(0, T; W_0^{1,2}(\mathcal{O})),$$
$$\varrho \in C_w([0, T]; L^\gamma(\mathcal{O})),$$
$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0, T; L^2(\mathcal{O})).$$

Continuity equation

Renormalized formulation of the continuity equation (DiPerna-Lions, '89):

ϱ satisfies for all $\psi \in C_c^\infty(Q)$

$$\int_Q b(\varrho)\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t = \int_Q b(\varrho)\mathbf{u}\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t - \int_Q \big(b'(\varrho)\varrho - b(\varrho)\big)\,\mathrm{div}\,\mathbf{u}\,\psi \,\mathrm{d}x\,\mathrm{d}t$$

- $b \in C^1(\mathbb{R})$ with $b'(z) = 0$ for all $z \geq M(b)$;
- ϱ solves weak formulation too $(b(z) = z)$.

Known results

- Existence of weak solutions for $\gamma \geq \frac{9}{5}$ by Lions (1998);
- Existence of weak solutions for $\gamma > \frac{3}{2}$ by Feireisl, Novotný, Petzeltová (2001);
- We have

$$\varrho \in L^{\frac{5}{3}\gamma-1}(Q), \quad \mathbf{u} \in L^2(0,T;W_0^{1,2}(\mathcal{O})),$$

$$\varrho\mathbf{u} \in C_w([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O})),$$

$$\varrho\mathbf{u}\otimes\mathbf{u} \in L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\mathcal{O})).$$

Stochastic Navier–Stokes equations

Velocity field $\mathbf{u}$ and density ϱ on $Q = (0, T) \times \mathcal{O}$

$$\begin{cases} \mathrm{d}(\varrho\mathbf{u}) = [\mu\Delta\mathbf{u} - \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - a\nabla\varrho^\gamma]\,\mathrm{d}t + \mathbb{G}\,\mathrm{d}W & \text{in } Q, \\ \mathrm{d}\varrho = -\operatorname{div}(\varrho\mathbf{u})\,\mathrm{d}t & \text{in } Q, \\ \varrho(0,\cdot) = \varrho_0 & \text{in } \mathcal{O}, \\ \varrho(0,\cdot)\mathbf{u}(0,\cdot) = \mathbf{q}_0 & \text{in } \mathcal{O}. \end{cases}$$

- Momentum equation in the weak sense: $\int_{\mathcal{O}} \varrho\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x = \dots$ for all $\boldsymbol{\varphi} \in C_c^\infty(\mathcal{O})$;
- Mass equation in the renormalized sense: $\mathrm{d}b(\varrho) = \dots$ for all $b \in C^1(\mathbb{R})$ with $b'(z) = 0$ for all $z \geq M_b$.

Concept of solution (1)

A finite energy weak martingale solution is a quantity

$$\big((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \varrho, \mathbf{u}, W\big).$$

- $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis,
- W is an $(\mathscr{F}_t)$-cylindrical Wiener process,
- $\varrho \geq 0$ is $(\mathscr{F}_t)$-adapted and $\varrho \in C_w([0, T]; L^\gamma(\mathcal{O}))$ $\mathbb{P}$-a.s.,
- $\mathbf{u}$ is a random variable and $\mathbf{u} \in L^2(0, T; W^{1,2}(\mathcal{O}))$ $\mathbb{P}$-a.s.,
- $\varrho\mathbf{u}$ is $(\mathscr{F}_t)$-adapted $\varrho\mathbf{u} \in C_w([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O}))$ $\mathbb{P}$-a.s.,
- $\Lambda = \mathbb{P} \circ \big(\varrho(0), \varrho\mathbf{u}(0)\big)^{-1}$.

Concept of solution (2)

- $(\varrho, \mathbf{u})$ solves the momentum equation in the weak sense $\mathbb{P}$-a.s.
- $(\varrho, \mathbf{u})$ solves the continuity equation in the renormalized sense $\mathbb{P}$-a.s.
- We have the energy inequality

$$\mathrm{d} \int_{\mathcal{O}} \left(\frac{1}{2} \varrho(t) \left| \mathbf{u}(t) \right|^2 + \frac{a}{\gamma - 1} \varrho^{\gamma}(t) \right) \mathrm{d}x \ \mathrm{d}t + \mu \int_{\mathcal{O}} |\nabla \mathbf{u}|^2 \, \mathrm{d}x$$
$$\leq \int_{\mathcal{O}} \mathbf{u} \cdot \mathbb{G}(\varrho, \varrho \mathbf{u}) \, \mathrm{d}W + \frac{1}{2} \sum_{k \geq 1} \int_{\mathcal{O}} \frac{|\mathbf{g}_k(\varrho, \varrho \mathbf{u})|^2}{\varrho} \ \mathrm{d}x \ \mathrm{d}t$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$.

Invariant measures (1)

- Statistical approach to fluid mechanics: is there an equilibrium state such that time-averages of the observable tend to this state as $t \to \infty$?
- Well-known for finite dimensional stochastic differential equations: invariant measure defined via transition semigroup

$$P_t(\varphi)(x) = \mathbb{E}\varphi(X_t(x)), \quad \varphi \in C_b(\mathbb{R}),$$

where (X_t) solves... with initial datum x.

- Define dual P_t^* on space of probability measures by

$$\int_{\mathbb{R}} \varphi \, \mathrm{d}P_t^* \nu = \int_{\mathbb{R}} P_t \varphi \, \mathrm{d}\nu \quad \forall \varphi \in C_b(\mathbb{R}), \, \nu \in \mathcal{M}(\mathbb{R}).$$

Invariant measures (2)

- A measure μ is called invariant if $P_t^* \nu = \nu$ or, equivalently
$$\int_{\mathbb{R}} P_t \varphi \, \mathrm{d}\nu = \int_{\mathbb{R}} \varphi \, \mathrm{d}\nu$$
$\Rightarrow$ probability distribution of X_t is independent of t.
- A stochastic process is stationary if its probability distribution is independent of time. Example: (W_t) Wiener process $\Rightarrow W(t+h) - W(t)$ is stationary for fixed h.
- If (X_t) is stationary solution to some SDE its probability law is an invariant measure.
- Semigroup property $P_{t+s} = P_t \circ P_s$ requires uniqueness!!

Stationarity

Definition (Stationary stochastic process)

Let $\mathbf{U} = \{\mathbf{U}(t); t \in [0, \infty)\}$ be an X-valued stochastic process. We say that $\mathbf{U}$ is *stationary* provided the joint laws

$$\mathcal{L}(\mathbf{U}(t_1 + \tau), \ldots, \mathbf{U}(t_n + \tau)), \quad \mathcal{L}(\mathbf{U}(t_1), \ldots, \mathbf{U}(t_n))$$

on X^n coincide for all $\tau \geq 0$, for all $t_1, \ldots, t_n \in [0, \infty)$.

- Here $\mathcal{L}$ denotes the law on X^n, i.e.

$$\mathcal{L}(Y_1, ..., Y_n)(B) = \mathbb{P}((Y_1, ..., Y_n) \in B) \quad B \subset X^n$$

for X-valued random variable $Y_1, ..., Y_n$.

Incompressible Navier–Stokes equations

- Existence of stationary martingale solutions by Flandoli-Gątarek 1995;
- $L^2(0,T;W^{1,2}(\mathcal{O}))$ becomes (almost) $L^\infty(0,T;W^{1,2}(\mathcal{O}))$ $\Rightarrow$ Stationary solutions are smooth (but depend on time)!!!
- Existence of unique invariant measure by Da Prato-Debussche 2003 using Kolmogorov equation (equation for $\mathbb{E}(\varphi(\mathbf{u}(t,x)))$, where $\varphi \in C_b(L^2)$).
- Note that $\mathbf{u} \in C_w([0,T];L^2(\mathcal{O}))$, so $\mathbf{u}(t) \in L^2(\mathcal{O})$ for ANY t.

Weak stationarity

Definition (Weakly stationary random variable)

Let $\mathbf{U} : \Omega \to \mathcal{D}'((0,\infty) \times \mathcal{O})$ be weakly measurable. Let $\mathcal{S}_\tau$ be the time shift on the space of trajectories given by $\mathcal{S}_\tau \varphi(t) = \varphi(t+\tau)$. We say that $\mathbf{U}$ is *weakly stationary* provided the laws

$$\mathcal{L}\left(\langle \mathbf{U}, \mathcal{S}_{-\tau}\varphi_1\rangle, \dots, \langle \mathbf{U}, \mathcal{S}_{-\tau}\varphi_n\rangle\right), \quad \mathcal{L}\left(\langle \mathbf{U}, \varphi_1\rangle, \dots, \langle \mathbf{U}, \varphi_n\rangle\right)$$

on $\mathbb{R}^n$ coincide for all $\tau \geq 0$, $\varphi_1, \dots, \varphi_n \in C_c^\infty((0,\infty) \times \mathcal{O})$.

- Here $\mathcal{L}$ denotes the law on $\mathbb{R}^n$, i.e.

$$\mathcal{L}(Y_1, ..., Y_n)(B) = \mathbb{P}((Y_1, ..., Y_n) \in B) \quad B \subset \mathbb{R}^n$$

for real valued random variable $Y_1, ..., Y_n$.

Theorem (Breit, Feireisl, Hofmanová, Maslowski, PTRF, '19)

Let the total mass be given by $M_0 \in (0, \infty)$, that is,

$$M_0 = \int_{\mathcal{O}} \varrho(t, x) \, \mathrm{d}x \quad \textit{for all} \quad t \in (0, \infty).$$

Then there exists a stationary finite energy weak martingale solution $[\varrho, \mathbf{u}, W]$ satisfying complete slip boundary conditions.

- More restrictive assumptions on noise:

$$\mathbb{G}(\varrho, \varrho\mathbf{u})\mathrm{d}W = \varrho\mathbb{F}(\varrho, \varrho\mathbf{u})\mathrm{d}W$$

  with $\mathbb{F}$ bounded.
- Extension to no-slip b.c. possible (Korn-Poincaré inequality needed).

Four layer approximation scheme

- existence of an invariant measure on the basic level by Krylov-Bogoliubov method:
  1. strong Feller property (continuous dependence on initial data)
  2. solution $(\varrho, \mathbf{u})$ is a Markov process
  3. tightness of

$$\left\{ \frac{1}{T} \int_0^T \mathcal{L}[\mathbf{u}(t)] \, \mathrm{d}t; \, T > 0 \right\}, \; \left\{ \frac{1}{T} \int_0^T \mathcal{L}[\varrho(t)] \, \mathrm{d}t; \, T > 0 \right\}.$$

- new global-in-time estimates needed
- stationarity preserved under limit procedures

Pressure law

Suppose that for some $\overline{\varrho} > 0$

$$p(\varrho) \approx \overline{p}(\overline{\varrho} - \varrho)^{-\beta}.$$

- By energy estimates control $P(\varrho) \in L^1$, where $P(\varrho) = \varrho \int_0^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z$;
- The singularity of the pressure at $\overline{\varrho}$ yields the (deterministic) bound $\varrho \leq \overline{\varrho}$;
- This hypothesis is relevant for any real fluid!

Bounded moments (1)

There are constants $\mathcal{E}_\infty(m)$, $m = 1, 2, \dots$ universal and independent of the initial condition s.t. we have

bounded moments of the total energy $E(\varrho, \mathbf{m}) = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)$

$$\limsup_{t\to\infty} \mathbb{E}\left(\int_{\mathcal{O}} E(\varrho, \varrho\mathbf{u})\,\mathrm{d}x \right)^m \leq \mathcal{E}_\infty(m),\ \ m = 1, 2, \dots$$

$\Rightarrow$**Asymptotic compactness.** The law of the time shifts of a fixed solution

$$\mathcal{L}[\varrho(\cdot + \tau_n), \mathbf{u}(\cdot + \tau_n)], \tau_n \to \infty$$

is tight in a suitable trajectory space.

Bounded moments (2)

- The energy inequality yields

$$\Big[\mathcal{E}(t)\Big]_{t=\tau_1}^{t=\tau_2} + \int_{\tau_1}^{\tau_2} \int_{\mathcal{O}} \mu |\nabla \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \dots;$$

- Dissipation beats energy s.t.

$$\Big[\mathcal{E}(t)\Big]_{t=\tau_1}^{t=\tau_2} + \int_{\tau_1}^{\tau_2} \int_{\mathcal{O}} \varrho |\mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \dots;$$

- Pressure estimates yield for $\mathcal{D} = \mathcal{E} - \varepsilon \int_{\mathcal{O}} \varrho \mathbf{u} \cdot \mathcal{B}[\varrho - \frac{M}{|Q|}] \, \mathrm{d}x$

$$\mathbb{E}|\mathcal{D}|^m(\tau) \leq \exp(-D_m \tau) \left(\mathbb{E}|\mathcal{D}(0)|^m - \frac{C_m}{D_m} \right) + \frac{C_m}{D_m};$$

- Replace $\mathcal{D}$ using $|\mathcal{D} - \mathcal{E}| \lesssim \sqrt{\mathcal{E}}$.

Stationary solutions à la Itô-Nisio

Theorem (Breit, Feireisl & Hofmanová, preprint)

Let $\left((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), \varrho, \mathbf{u}, W\right)$ be a dissipative martingale solution such that $\mathbb{E}\mathcal{E}(0)^4 < \infty$,.... Then there is a sequence $T_n \to \infty$ and a stationary solution

$$\left((\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{W}\right),$$

with $\mathcal{L}\left[\mathcal{S}_\tau[\varrho, \mathbf{u}, W]\right] = \mathcal{L}[\varrho, \mathbf{u}, W]$ such that

$$\frac{1}{T_n}\int_0^{T_n} \mathcal{L}\left[\mathcal{S}_t\left[\varrho, \mathbf{u}, W\right]\right] \mathrm{d}t \to \mathcal{L}\left[\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{W}\right] \text{ as } n \to \infty.$$

Navier–Stokes–Fourier equations

- Heat-conducting fluids depending on temperature ϑ;
- Add entropy balance for specific entropy s

$$\mathrm{d}(\varrho s) + \left[\mathrm{div}(\varrho s \mathbf{u}) - \mathrm{div}\left(\frac{\kappa \nabla \vartheta}{\vartheta}\right)\right] \mathrm{d}t = \frac{1}{\vartheta}\left(\mu |\nabla \mathbf{u}|^2 + \kappa \frac{|\nabla \vartheta|^2}{\vartheta}\right) \mathrm{d}t$$

  or equivalent equation for internal energy e, where

$$p(\varrho, \vartheta) = \varrho^{\frac{5}{3}} + \varrho\vartheta + \tfrac{a}{3}\vartheta^4, \;\; e(\varrho, \vartheta) = \tfrac{3}{2}\varrho^{\frac{2}{3}} + c_v \vartheta + a\frac{\vartheta^4}{\varrho},$$
$$s(\varrho, \vartheta) = \tfrac{4a}{3}\frac{\vartheta^3}{\varrho} + \log(\vartheta^{c_v}) - \log(\varrho).$$

- Existence of weak martingale solutions Breit & Feireisl, IUMJ, ‘20.

Stationary solutions

Theorem (Breit & Feireisl, IUMJ (2020))

There are no stationary solutions to the stochastic Navier–Stokes–Fourier equations.

Energetically closed system with energy equality

$$\mathrm{d} \int_{\mathcal{O}} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] = \int_{\mathcal{O}} \mathbb{G} \cdot \mathbf{u} \, \mathrm{d}W + \sum_{k \geq 1} \int_{\mathcal{O}} \frac{1}{2} \varrho^{-1} |\mathbf{g}_k|^2 \, \mathrm{d}t.$$

Theorem (Breit, Feireisl & Hofmanová, Math. Ann., in press)

There is a stationary solution to the stochastic Navier–Stokes–Fourier equation with non-homogeneous Neumann-boundary values: $\partial_\nu \vartheta|_{\partial \mathcal{O}} = \vartheta - \Theta_0$.

D. Breit & E. Feireisl: *Stochastic Navier–Stokes–Fourier equations.* **Indiana Univ. Math. J.** 69, 911–975. (2020)

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