The motion of generalized Newtonian fluids

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Outline

1. Introduction to fluid mechanics
2. Stationary problems
3. Parabolic problems
4. Further constitutive laws

Navier-Stokes equations

Find a velocity field $\mathbf{v} : Q \to \mathbb{R}^d$ and a pressure function $\pi : Q \to \mathbb{R}$ satisfying the following

system of partial differential equations

$$\begin{cases} -\partial_t \mathbf{v} + \operatorname{div} \boldsymbol{\sigma} = (\nabla \mathbf{v})\mathbf{v} + \nabla \pi - \mathbf{f} & \text{on } Q, \\ \operatorname{div} \mathbf{v} = 0 & \text{on } Q, \\ \mathbf{v} = 0 & \text{on } \partial\Omega, \\ \mathbf{v}(0, \cdot) = \mathbf{v}_0 & \text{on } \Omega, \end{cases}$$

- $Q := \Omega \times (0, T)$ with $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ and $T > 0$;
- $\mathbf{f} : Q \to \mathbb{R}^d$ is a system of volume forces;
- $\boldsymbol{\sigma} : Q \to \mathbb{R}^{d \times d}$ is the stress deviator.

Constitutive law

In order to characterize the specific fluid under consideration we need a constitutive law, which relates $\boldsymbol{\sigma}$ and the symmetric gradient

$$\varepsilon(\mathbf{v}) := \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right).$$

- Newtonian fluid: $\boldsymbol{\sigma} = \nu\varepsilon(\mathbf{v})$ (water, air and the most oils);
- Generalized Newtonian fluid: $\boldsymbol{\sigma} = \nu(|\varepsilon(\mathbf{v})|)\varepsilon(\mathbf{v})$;
- the viscosity ν is a function of the shear rate $|\varepsilon(\mathbf{v})|$;
- ν increasing $\Rightarrow$ shear thickening (batter);
- ν decreasing $\Rightarrow$ shear thinning (blood, ketchup).

Power law model

Most popular model among rheologists

for $1 < p < \infty$ and $\nu_0 > 0$

$$\nu(|\varepsilon(\mathbf{v})|) = \nu_0|\varepsilon(\mathbf{v})|^{p-2},$$
$$\nu(|\varepsilon(\mathbf{v})|) = \nu_0(1 + |\varepsilon(\mathbf{v})|)^{p-2}.$$

- $p > 2 \Rightarrow$ shear thickening fluid (batter);
- $p < 2 \Rightarrow$ shear thinning fluid (blood, ketchup);
- $p = 2 \Rightarrow$ Newtonian fluid.

Mathematical questions

- In which function spaces do we have existence of solutions?
- Under which assumptions is the solution unique?
- How are the regularity properties of solutions?

The p-Stokes problem (1)

Find $\mathbf{v} : \Omega \to \mathbb{R}^d$ and $\pi : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} \operatorname{div}\left(|\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v})\right) = \nabla\pi - \mathbf{f} & \text{on } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{on } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega. \end{cases}$$

- Function space for $\mathbf{v}$:

$$\mathring{W}^{1,p}_{\operatorname{div}}(\Omega,\mathbb{R}^d) := \left\{\mathbf{u} \in W^{1,p}(\Omega,\mathbb{R}^d) : \ \mathbf{u}|_{\partial\Omega} = 0, \ \operatorname{div}\mathbf{u} = 0\right\};$$

- Function space for π:

$$L^{p'}_0(\Omega) := \left\{u \in L^{p'}(\Omega) : \ \int_\Omega u\,dx = 0\right\}.$$

The p-Stokes problem (2)

Minimize the functional

$$\mathcal{J}[\mathbf{v}] := \frac{1}{p}\int_\Omega |\varepsilon(\mathbf{v})|^p\,dx - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx$$

in the space $\mathring{W}^{1,p}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$.

- Existence theory via direct method using Korn's inequality

$$\int_\Omega |\nabla\mathbf{v}|^p\,dx \le c\int_\Omega |\varepsilon(\mathbf{v})|^p\,dx$$

for all $\mathbf{v}\in\mathring{W}^{1,p}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$.

The p-Stokes problem (3)

Minimizer is weak solution

$$\int_\Omega |\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v}) : \varepsilon(\varphi)\,dx = \int_\Omega \mathbf{f}\cdot\varphi\,dx, \quad \varphi \in C^\infty_{0,\mathrm{div}}(\Omega,\mathbb{R}^d).$$

- Reconstruction of the pressure π using solutions to

$$\operatorname{div}\mathbf{F} = \mathbf{f} \quad \Rightarrow \quad \mathbf{F} = \mathrm{Bog}(\mathbf{f}),$$
$$\mathrm{Bog} : L^p_0(\Omega) \to \mathring{W}^{1,p}(\Omega,\mathbb{R}^d).$$

The p-Stokes problem (4)

Regularity

$$V := |\varepsilon(\mathbf{v})|^{\frac{p}{2}} \in W^{1,2}(\Omega);$$
$$\mathbf{v} \in C^{1,\alpha}(\Omega_0, \mathbb{R}^d), \quad \mathcal{L}^d(\Omega \setminus \Omega_0) = 0;$$
$$\mathbf{v} \in C^{1,\alpha}(\Omega, \mathbb{R}^2) \quad \text{if} \quad d = 2.$$

- Results by Fuchs in 1996, later by Naumann;
- 2D: Kaplický, Málek and Stará in 1999.

The stationary p-Navier-Stokes problem (1)

Existence theory for stationary generalized Newtonian fluids.

Equation of motion

$$\operatorname{div}\left(|\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v})\right) = \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla\pi - \mathbf{f}$$

- No variational approach available;
- Consider an approximated system whose solution $\mathbf{v}^n$ is known to exist together with

$$\mathbf{v}^n \rightharpoonup: \mathbf{v} \quad \text{in} \quad W^{1,p}(\Omega,\mathbb{R}^d).$$

The stationary p-Navier-Stokes problem (2)

Convergence of the convective term

$$\int_\Omega \mathbf{v}^n \otimes \mathbf{v}^n : \nabla\varphi \, dx \longrightarrow \int_\Omega \mathbf{v} \otimes \mathbf{v} : \nabla\varphi \, dx$$

- Compact embedding

$$W^{1,p}(\Omega, \mathbb{R}^d) \hookrightarrow L^2(\Omega, \mathbb{R}^d), \quad p > \frac{2d}{d+2}.$$

The stationary p-Navier-Stokes problem (3)

Energy convergence

$$\int_\Omega |\varepsilon(\mathbf{v}^n)|^{p-2}\varepsilon(\mathbf{v}^n) : \varepsilon(\varphi)\, dx \longrightarrow \int_\Omega |\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v}) : \varepsilon(\varphi)\, dx$$

- Almost everywhere-convergence $\varepsilon(\mathbf{v}^n) \to \varepsilon(\mathbf{v})$.
- Monotone-operator theory:

$$\int_\Omega \big(\mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(v))\big) : \varepsilon\left(\mathbf{v}^n - \mathbf{v}\right)\, dx \longrightarrow 0,$$

$$\big(\mathbf{S}(\zeta) - \mathbf{S}(\xi)\big) : (\zeta - \xi) > 0 \quad \text{if} \quad \zeta \neq \xi.$$

The stationary p-Navier-Stokes problem (4)

Test the equation by

$$\mathbf{u}^n = \mathbf{v}^n - \mathbf{v}.$$

- Standard if $p > \frac{9}{5}$ many interesting fluids are between $[\frac{3}{2}, 2]$;
- L^∞-truncation if $p \geq \frac{3}{2}$ by Frehse, Málek, Steinhauer in 1997:

$$\mathbf{u}_\lambda = \mathbf{u} \quad \text{on} \quad \{x : \ |\mathbf{u}(x)| \leq \lambda\}, \quad \|\mathbf{u}_\lambda\|_\infty \leq \lambda.$$

- For blood we have $p \approx 1.21$.

Lipschitz truncation (1)

One can define the Lipschitz truncation

of a Sobolev function $\mathbf{u}$ by

$$T^{\lambda}\mathbf{u} := \begin{cases} \mathbf{u} & , \quad \text{on } [M(|\nabla \mathbf{u}|) \leq \lambda] \\ \sum_j \varphi_j \mathbf{u}_j & , \quad \text{on } [M(|\nabla \mathbf{u}|) > \lambda] \end{cases}$$

where $M : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is the Hardy-Littlewood maximal function

$$(Mf)(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy.$$

Lipschitz truncation (2)

- If $\mathbf{u} \in \mathring{W}^{1,p}$ then $T^{\lambda}\mathbf{u} \in \mathring{W}^{1,\infty}$;
- Lipschitz truncation of Sobolev-functions goes back to Acerbi and Fusco (1988);
- Firstly used in fluid mechanics by Frehse, Málek, Steinhauer in 2003;
- Advanced by Diening, Málek, Steinhauer in 2006;
- Existence theory for the stationary p-Navier-Stokes problem provided

$$p > \frac{6}{5}.$$

The non-stationary p-Navier-Stokes problem

Existence theory for non-stationary generalized Newtonian fluids.

Equation of motion

$$-\partial_t \mathbf{v} + \operatorname{div}\left(|\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v})\right) = \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla\pi - \mathbf{f}$$

Weak formulation: for all $\varphi \in C^{\infty}_{0,\mathrm{div}}((-\infty, T) \times \Omega)$

$$\int_Q |\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v}) : \varepsilon(\varphi)\, dx\, dt = \int_Q \mathbf{v} \otimes \mathbf{v} : \nabla\varphi\, dx\, dt + \int_Q f \cdot \varphi\, dx\, dt$$
$$+ \int_Q \mathbf{v}\, \partial_t\varphi\, dx\, dt + \int_\Omega \mathbf{v}_0\, \varphi(0)\, dx.$$

Existence-theory (1)

Function space

$$\mathbf{v} \in L^p(0, T; \mathring{W}^{1,p}_{\mathrm{div}}(\Omega, \mathbb{R}^d)) \cap L^2(Q, \mathbb{R}^d),$$
$$\partial_t \mathbf{v} \in L^\sigma(0, T; \mathring{W}^{-1,\sigma}(\Omega, \mathbb{R}^d)), \quad \sigma > 1.$$

- Compactness of $\mathbf{v}^n$ in L^2 by Aubert-Lions;
- Monotone-operator theory provided $p > \frac{11}{5}$;
- L^∞-truncation provided $p > \frac{8}{5}$ by Wolf in 2007;
- Lipschitz truncation provided $p > \frac{6}{5}$ by Diening, Ruzicka, Wolf in 2010.

Existence-theory (2)

Parabolic scaling

$$Q_r^\alpha = (-\alpha r^2, \alpha r^2) \times B_r, \quad \alpha = \lambda^{2-p}.$$

- Decomposition of Q by means of cubes $(Q_{r_i}^\alpha)_{i\in\mathbb{N}}$;
- Lipschitz truncation $\mathbf{v}^\lambda$ with $\|\nabla\mathbf{v}^\lambda\|_\infty \leq c\lambda$;
- $\partial_t\mathbf{v}$ is connected with π;
- pressure decomposition $\pi = \pi_h + \pi_S + \pi_c$ via singular integrals in L^p-setting.

Open problems

- Convective term $\mathbf{v} \otimes \mathbf{v}$ always defined;
- Existence theory for $1 < p \leq \frac{6}{5}$;
- Regularity results if $\frac{12}{5} < p < \frac{10}{3}$ by Seregin in 1999;
- Millenium problem for $p = 2$.

Electro-rheological fluids (1)

The fluid reacts on an electric field modeled by

$p : (0, T) \times \Omega \to (1, \infty)$

$$\boldsymbol{\sigma} = \nu(t, x, |\varepsilon(\mathbf{v})|)\varepsilon(\mathbf{v}), \quad \nu(t, x, |\varepsilon(\mathbf{v})|) = \nu_0 |\varepsilon(\mathbf{v})|^{p(t,x)-2}.$$

- ν increases in 1ms for the factor 1000;
- Controlling of fluid properties without mechanical interaction;
- Many technological applications: actuators, clutches, shock absorbers, rehabilitation equipment;
- Firstly oberved by Winslow in 1949.

Electro-rheological fluids (2)

Generalized Lebesgue- and Sobolev-spaces

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\},$$
$$W^{1,p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \in L^{p(\cdot)}(\Omega),\ \nabla u \in L^{p(\cdot)}(\Omega, \mathbb{R}^d) \right\}.$$

- $L^{p(\cdot)}(\Omega)$ is a Banach-space via

$$\|u\|_{p(\cdot)} := \inf \left\{ k : \int_\Omega \left| \frac{u(x)}{k} \right|^{p(x)} dx \le 1 \right\}.$$

Electro-rheological fluids (3)

Existence of weak solutions in the steady case provided

$p : \Omega \to (1, \infty)$ is Hölder continuous and

$$\inf_{\Omega} p > \frac{2d}{d+2}.$$

- Diening, Malék and Steinhauer in 2006 via Lipschitz truncation;
- Study of $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ by Diening and Ruzicka: continuity of maximal function, smooth approximation, singular integral operators, Korn's inequality.

Electro-rheological fluids (4)

The non-stationary case:

- Current project: Breit, Diening, Ruzicka, Schwarzacher;
- reconstruction of the pressure fails $\rightsquigarrow$ solenoidal Lischitz truncation;
- problems with parabolic scaling $\alpha = \lambda^{p-2}$;
- no approach via Bochner spaces like $L^p(0, T; W^{1,p})$.

Prandtl-Eyring fluids (1)

Eyring obtained in 1936

$$\boldsymbol{\sigma} = \nu(|\varepsilon(\mathbf{v})|)\varepsilon(\mathbf{v}), \quad \nu(|\varepsilon(\mathbf{v})|) = \nu_0 \frac{\ln(1 + |\varepsilon(\mathbf{v})|)}{|\varepsilon(\mathbf{v})|}.$$

- Very shear thinng $\rightsquigarrow$ lubricants;
- Consideration of Stokes-problems by Fuchs and Seregin in 1999;
- Stationary Navier-Stokes problem in 2D by Breit, Diening and Fuchs in 2011.

Prandtl-Eyring fluids (2)

Funcation space for $h(t) = t\ln(1+t)$

$$V^{1,h}(\Omega) := \left\{ \mathbf{w} \in L^1(\Omega, \mathbb{R}^d) : \int_\Omega h(|\varepsilon(\mathbf{w})|)\, dx < \infty \right\}.$$

- $V^{1,h}(\Omega) \hookrightarrow L^{d/(d-1)}(\Omega, \mathbb{R}^d)$;
- Korn's inequality does not hold (Breit and Diening, 2011);
- pressure reconstruction fails $\rightsquigarrow$ solenoidal Lischitz truncation;
- Maximal function is not continuous,
- Parabolic problem is still open.

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