SPP2410 Workshop: Mathematics of Compressible Fluids – Analysis and Numerics

Book of Abstracts

Contents

Invited Speakers

Numerical Conservation and Its Challenges: Non-Conservative Terms, Structure-Preserving Schemes, and More

Rémi Abgrall*

The concept of conservation in conservation laws

In this first talk, I will review the classical concepts of conservation for non linear hyperbolic problems: weak entropy solutions, and how this is translated at the discrete level, via the Lax-Wendroff theorem. Then I will show that all known schemes can be put in a form where the conservation principles are not written at the level of cells or elements interfaces, but at the level of elements. This concept is more flexible, though equivalent in some sense. I will first show what I mean by equivalent and how the flexibility can be used to accommodate additional conservation laws, and even entropy preservation or entropy dissipation. In particular I will show how to systematically modify a known scheme to accommodate a family of additional conservation laws. For example, this can be used to get scheme that also preserve the kinetic momentum, and additional constraints such as kinetic energy "preservation".

Discretization of non-conservative terms, potential pitfalls, and possible solutions

In this second lecture, I will discuss the problems coming from the existence of non conservative terms in an hyperbolic problem. I will show on several examples arising for multiphase flow problems, how one can deal with this, and what are the limits of this game, at least in my opinion. I will also show how to build schemes, where starting from a non conservative formulation of a conservative problem, one nevertheless gets the right solution. An example is given for the Euler equation where the PDE system is written for the mass, velocity and the pressure. More complex systems will also be discussed.

An overview of structure-preserving schemes

In this last lecture, I will discuss the concepts of structure preserving schemes. Structure preservation can occur in several cases: preserving the invariant domain of a PDE, preserving involutions (for example in acoustics or MHD). In these examples, I will show practical solutions, and I will also try to discuss some open problems.

*University of Zurich

Dissipative solutions to models of inviscid fluids

Eduard Feireisl*

Models of inviscid or perfect fluids in continuum fluid mechanics are based on the Euler system. We consider the general setting of the Euler system of gas dynamics.

The first lecture is the introduction to the problems and a review of the state-of-the art. We recall the recent results on well/ill posedness of the Euler system obtained by the method of convex integration as well as results on blow-up of smooth solutions.

In the second lecture, we identify the limit of consistent approximations of the Euler system - the dissipative measure-valued (DMV) solutions. We show that this concept is convenient to recover uniqueness for the Euler system.

In the third lecture, we show that the measure–valued solutions of the Euler system of gas dynamics generated by oscillatory sequences of consistent approximations violate the principle of maximal entropy production formulated by Dafermos. There are numerical results illustrating that solutions obtained by standard numerical methods may be oscillatory and thus do not comply with the Dafermos criterion.

*Czech Academy of Sciences

Path-by-path regularization by noise for scalar conservation laws

Benjamin Gess*

We analyze the regularity-enhancing effects of stochastic perturbations in the setting of (nonlinear) scalar conservation laws. By leveraging the increased dispersive properties induced by stochasticity, we establish a pathwise regularization-by-noise result for scalar conservation laws. In particular, these results demonstrate the regularizing influence of fractional Brownian motion as a driving noise and extend the corresponding findings of [G., Souganidis; (2017)]. Furthermore, we provide a precise characterization of pathwise properties that lead to improved regularity. Specifically, we show that (ρ, γ)-regularity constitutes a sufficient pathwise condition ensuring such regularization effects. Additionally, we introduce a novel pathwise scaling property, which is likewise proven to be a sufficient condition for regularization.

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Convex limiting techniques for hyperbolic conservation and balance laws

Dmitri Kuzmin*

This talk is a review of modern algebraic flux correction tools for finite element discretizations of nonlinear hyperbolic problems. In particular, we present monolithic convex limiting (MCL) techniques for generic scalar equations, for the Euler equations of gas dynamics, and for the shallow water equations (SWE) with non-flat bathymetry [1, 2, 3]. The validity of discrete maximum principles is enforced using representation of spatial semi-discretizations in terms of admissible intermediate states [1, 4]. If necessary, additional limiting is performed to achieve entropy stability [5, 6]. In our MCL schemes for systems, problem-dependent inequality constraints are imposed on scalar functions of conserved variables to ensure physical and numerical admissibility of approximate solutions. The MCL discretization of the SWE system is positivity preserving, entropy stable, and well balanced [3]. We prove the claimed properties of our flux-corrected schemes and discuss relevant implementation details. The main focus of the talk is on continuous Galerkin methods and piecewise-linear approximations, but convex limiting is also an option for discontinuous Galerkin and finite volume approaches [7, 8, 9, 10]. Existing extensions to high-order (Bernstein or spectral) finite elements [5, 8, 9] and general Runge-Kutta methods [10] demonstrate the wide applicability of the MCL methodology.

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Random compressible flows

Mária Lukáčová-Medviďová*

Mathematical models arising in science and engineering inherit several sources of uncertainties. To predict reliable results, deterministic models are insufficient. In computational fluid dynamics, stochastic collocation or the Monte Carlo methods are typically used to quantify the propagation of data uncertainty. Despite the large popularity of these methods, their rigorous convergence analysis for compressible fluid flows was missing in general. In this talk, we will review our recent results obtained for the random compressible Euler and Navier-Stokes systems. We suppose that the initial and boundary data as well as model parameters, such as the viscosity coefficients, are random variables. Consequently, a solution of the PDE system will be a random process. The stochastic collocation or the Monte Carlo methods are combined with a suitable deterministic discretization scheme, such as a finite volume method. Since the compressible Navier-Stokes and the Euler equations are not uniquely solvable in the class of global weak solutions, we cannot apply pathwise arguments to analyze the random equations. Instead, we apply intrinsic stochastic compactness arguments via the Skorokhod representation theorem and the Gyöngy-Krylov method. We study both the statistical convergence rates as well as the approximation errors. The convergence of the deterministic Navier-Stokes or Euler system is realized via dissipative solutions. Numerical experiments will illustrate theoretical results.

*JGU Mainz

Compressible Euler System with Nonlocal Pressure

Raphael Danchin[*], Piotr Bogusław Mucha[†]

I will discuss a modification of the compressible barotropic Euler system with friction, where the conventional pressure term is replaced by a fuzzy nonlocal pressure. This nonlocal term is parameterized by $\varepsilon > 0$ and formally converges to the classical pressure as ε approaches zero. The primary challenge lies in demonstrating that this modified system serves as a reliable approximation of the classical compressible Euler system. We establish the global existence and uniqueness of regular solutions in the vicinity of the static state, characterized by a density of 1 and zero velocity. These results are proven independently of the parameter ε, allowing us to demonstrate the convergence of solutions to those of the classical Euler system. Additionally, we rigorously justify the convergence of the mass equation to various forms of the porous media equation in the asymptotic limit where friction tends to infinity. Notably, our results are derived in the context of the whole space, necessitating the use of Besov spaces.

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Modeling compressible two-phase flows with finite-time relaxation in the Baer-Nunziato model

Barbara Re*

Compressible multi-component and multi-phase flows are ubiquitous in technological applications and can exhibit a variety of topologies, such as sprays, bubbly flows, and dispersed flows. A key feature of these flows is the presence of dynamic interfaces that separate immiscible components. For numerical simulation, diffuse interface methods offer an effective strategy to handle these interfaces and the abrupt variations in thermochemical properties across them.

In this context, the Baer-Nunziato model and its variants describe two-phase flows through a system of seven equations (in 1D): mass, momentum, and energy balance for each phase, plus an evolution equation for the volume fraction [1]. This set of partial differential equations (PDEs) can be equipped with various relaxation terms to model how phasic variables approach equilibrium at interfaces.

When it is reasonable to assume instantaneous mechanical equilibrium between the phases, the equations are solved with pressure and velocity relaxation parameters tending to infinity. When non-equilibrium effects are of interest, finite relaxation parameters can be used [2]. Additional relaxation terms can also be introduced to model mass and energy transfer.

This talk discusses the full disequilibrium Baer-Nunziato two-phase model equipped with mechanical and thermo-chemical relaxation terms characterized by finite time scales [3]. This model is solved using operator splitting between the hyperbolic and relaxation parts, and this talk focuses on the latter. We present a time-accurate and efficient solver for simultaneous mechanical and thermo-chemical relaxation, applicable to generic equations of state. The successful simulation of two-phase flows involving condensation, evaporation, and mixed conditions are shown to illustrate the model's capabilities.

Acknowledgements

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Measure-valued solutions of the compressible Euler equations

Emil Wiedemann*

Inviscid fluids, as modelled by the Euler equations, do not allow for a satisfactory well-posedness theory at low regularity, which is believed to be relevant for the study of turbulence. Therefore, from the 1980s onwards, several related notions of "very weak" solutions have been investigated, such as measure-valued or dissipative. In recent years the topic has attracted more and more attention. I will give a motivation and an introduction to the topic and then present some current results (joint with D. Gallenmüller and with C. Klingenberg & S. Markfelder) that focus on the question of selection.

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Contributed Talks

A numerical scheme for compressible two-phase flows at all Mach numbers

Beatrice Battisti*, Walter Boscheri†

Many natural phenomena, such as volcanic eruptions, involve complex multi-phase flows. These flows often feature a mix of compressible and incompressible behaviors, making their modeling particularly challenging. As a result, no universally accepted model exists in the literature. However, most approaches are built upon the Baer-Nunziato equations for compressible two-phase flow [1].

We present a semi-implicit solver for the solution of the Baer-Nunziato model [2]. A novel linearly implicit discretization is proposed for both the pressure fluxes and the relaxation source terms, while the nonlinear convective terms are treated explicitly. This formulation leads to a CFL-type stability condition on the maximum admissible time step only based on the mean flow velocity, rather than on the sound speed of each phase, so that the novel scheme works uniformly for all Mach numbers.

Central finite difference operators on Cartesian grids are adopted for the implicit terms, thus avoiding artificial numerical diffusion that might destroy accuracy in the low Mach number regime. To comply with high Mach number flows, shock capturing finite volume schemes are employed for the approximation of the convective fluxes. The discretization of the nonconservative terms ensures the preservation of moving equilibrium solutions, making the new method well-balanced. The new scheme is also proven to be asymptotic preserving in the low Mach limit of the mixture model. Second order of accuracy in both space and time is achieved by means of an implicit-explicit (IMEX) time stepping algorithm combined with a total variation diminishing (TVD) reconstruction technique.

The proposed method is validated through a series of benchmark problems spanning a wide range of Mach numbers, demonstrating both its accuracy and robustness.

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Hyperbolic stochastic Galerkin formulations

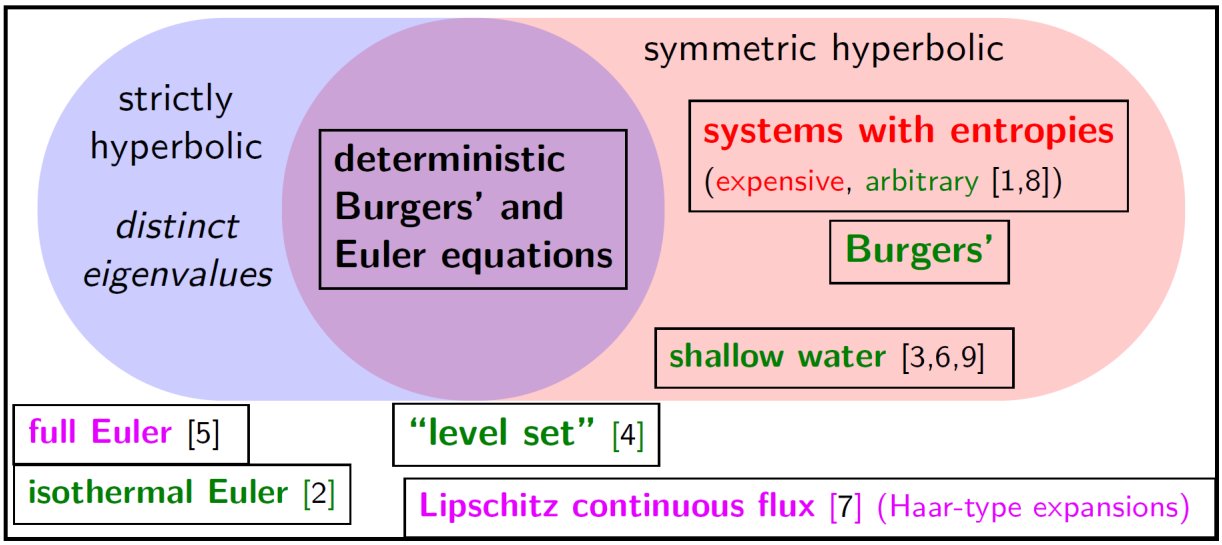
Stephan Gerster *

Stochastic Galerkin formulations to random hyperbolic equations are based on the idea that the functional dependence on the stochastic input is described a priori by a polynomial chaos expansion and a Galerkin projection is used to obtain deterministic evolution equations for the coefficients in the series. Then, all envolved mathematical operations, e.g. products and norms, must be adopted and applied to the variables in the governing equations. In general, results for hyperbolic systems are not available, since desired properties like hyperbolicity and the existence of entropies are not transferred to the intrusive formulation.

To this end, auxiliary variables have been introduced to establish wellposedness results. For instance, entropy-entropy flux pairs can be obtained by an expansion in entropy variables, i.e. the gradient of the deterministic entropy [1, 8]. Roe variables, which include the square root of the density, preserve hyperbolicity for Euler equations [2, 5]. The drawback of introducing auxiliary variables is an additional computational overhead that arises from an optimization problem, which is required to calculate the auxiliary variables.

These results exploit quadratic relationships that are expressed efficiently by the Galerkin product. Extensions of the classical polynomial chaos expansions to general nonlinearities, e.g. with Legendre or Hermite polynomials, however, are not straightforward. More general nonlinearities, which occur for instance in isentropic Euler, level-set equations or when including source terms in shallow water equations [9], can be expressed by wavelet families that are generated by Haar-type matrices [7]. Those are widely used in signal processing, e.g. for the Hadamard, Walsh, Chebyshev matrices and for the discrete cosine transform.

In this talk we give an overview on methods to circumvent the loss of hyperbolicity and discuss briefly numerical challenges.



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Statistical conservation laws for scalar model problems

Qian Huang [*], Christian Rohde [†]

It has been known for decades that the probability density functions (PDFs) for solutions of the random/stochastic incompressible Navier-Stokes equations can be represented by a hierarchy of linear equations, but an analogue for compressible flows is still lacking. Moreover, due to the high dimensionality of multipoint PDFs, it is crucial to develop numerical approximations for truncated versions of these hierarchies. As a first step towards this goal, we consider the viscous scalar conservation law

$$\partial_t u + \partial_x g(u) = \epsilon \partial_{xx} u, \quad t > 0, \quad x \in \mathbb{R}^d, \tag{1}$$

where $u = u(t, x, \xi)$ has random initial data $u_0(x, \xi)$, ξ is a random variable and $\epsilon > 0$ is a constant. This model problem mimics, at the scalar level, the interplay of nonlinear hyperbolic transport and viscosity operators that plays a central role for understanding turbulent flows. For this problem, we derive several new types of hierarchies of linear equations for the PDFs of the solution (one of which is the analogue to the incompressible NSE counterpart). The key difference is that, unlike incompressible flows (with the divergence-free condition), the convection term is nonlocal for the PDF master equations in the compressible-like cases. An important observation is that the cumulative distribution function (CDF) of the solution can avoid this nonlocality. The one-point CDF $F = F(t, x, v)$ satisfies

$$\partial_t F + g'(v)\partial_x F + \epsilon m(t, x, v)\partial_v F = 0, \quad F(0, t, x) = G(x, v), \tag{2}$$

where $m(t, x, v)$ is manifestation of the *unresolved* viscous effects that generate the hierarchies. We show that this system exhibits a 'dissipative anomaly': under certain conditions with nonlinear flux $g(v)$, the term $\epsilon m(t, x, v)$ does not vanish as $\epsilon \to 0$. We further demonstrate that one can approximate $m(t, x, v)$ with sampled solutions to (1) and then use the approximated m to solve (2) for the CDFs (and PDFs). The feasibility of this approach is supported both theoretically and numerically. Figure 1 illustrates the resulting PDFs and shows the error reduction as the number of samples increases.

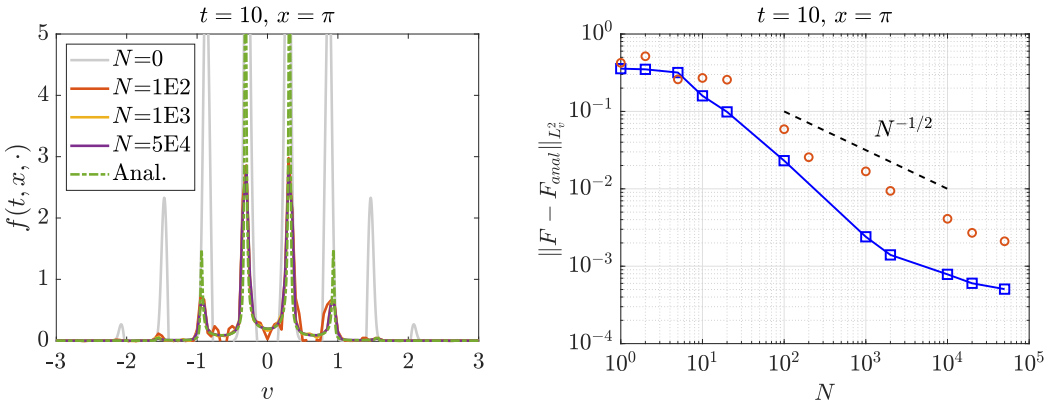

Figure 1: Approximated PDFs (left) and reduced errors (right) with increasing number of samples for a specific random initial data $u_0(x, \xi) = \sin x + \xi$ with $x \in \mathbb{R}/[0, 2\pi)$, $\xi \sim \mathcal{N}(0, 0.5)$ and $g(u) = u^2/2$.

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A Relaxation Formulation of the Navier-Stokes-Korteweg Equations

Jens Keim [*], Claus-Dieter Munz[*], Christian Rohde [†]

Multiphase flows with phase transition on the pore-scale are a growing and challenging field of research. Diffuse interface approaches are promising candidates for this task due to their ability to deal with two-phase flow, surface tension, phase transition and complex domain topologies. A well known model which incorporates all the aforementioned physical effects is the Navier-Stokes-Korteweg-van der Waals (NSK) model. However, several issues have to be faced if the numerical discretization of the NSK system is aimed. First, the non-convex bulk Helmholtz free energy induces a non-monotonous pressure function, which in turn results in mixed hyperbolic-elliptic first-order fluxes. Secondly, the gradient dependence of the underlying energy potential generates a third-order derivative in the momentum equation, which prevents the straight forward use of boundary conditions at the three-phase contact line.
In this talk, we present the non-isothermal extension of our recently presented relaxation formulation of the isothermal NSK equations [1, 2]. The model is consistent with the Second Law of Thermodynamics and converges to the non-isothermal NSK equations for an appropriate choice of the modeling parameters. Moreover, strict hyperbolicity of the convective fluxes is guaranteed for a sufficiently large Korteweg parameter. Furthermore, we can provide thermodynamic consistent non-equilibrium boundary conditions for the fluid-solid interface, which can be specified without the requirement to solve a global elliptic equation [3] or any additional iterative procedure at the solid boundary [4].
For the numerical discretization we utilize a high-order discontinuous Galerkin scheme. Our numerical experiments demonstrate a very good agreement with the original NSK model in several space dimensions. The comparison is concluded by the simulation of highly dynamic 3D droplet collisions. Turning to the pore-scale, we proof our model to be consistent with the Young-Laplace law. Finally, we demonstrate the ability of our framework to handle complex domains by the simulation of a spinodal decomposition in a synthetic porous medium.

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On conditional convergence rates of finite volume methods for hyperbolic systems in 1D

Fabio Leotta*

Recently, Bressan et al. [1] have proposed an a-posteriori error estimation framework for abstract numerical methods for hyperbolic systems in 1D: Assuming that the numerical solution - at the discretization level (τ, h) in question - is (a) small in BV, and (b) does not oscillate too much outside a finite number of narrow strips in $[0, T] \times \mathbb{R}$, they prove that if the numerical method is consistent and entropy stable in a $(W^{1,\infty})^*$-sense, then an a-posteriori error estimate can be derived. However, if (a) and (b) hold uniformly in the discretization parameters, the a-posteriori bound turns into an a-priori error estimate. In this sense, one can speak of a conditional convergence result for numerical schemes that yields an explicit convergence rate of $h^{1/3}$.

Building on the above, we discuss consistency and entropy stability of forward Euler finite volume methods in the $(W^{1,\infty})^*$-sense. In addition, we will consider a two-stage P^1-RKDG method with abstract slope limiting.

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*TU Darmstadt

A well-balanced PAMPA method for shallow water equations with horizontal temperature gradients on unstructured triangular meshes

Yongle Liu*

In this talk, we introduce a novel well-balanced Point-Average-Moment PolynomiAl-interpreted (PAMPA) numerical method for solving the two-dimensional shallow water equations with temperature gradients on unstructured triangular meshes. The proposed PAMPA method use a globally continuous representation of the variables, with degree of freedoms (DoFs) consisting of point values on the edges and average values within each triangular element. The update of cell averages is carried out using a conservative form of the partial differential equations (PDEs), while the update of point values—unconstrained by local conservation—follows a non-conservative formulation. The powerful PAMPA framework offers great flexibility in the choice of variables for the non-conservative form, including conservative variables, primitive variables, and other possible sets of variables. In order to preserve a wider class of steady-state solutions, we introduce pressure-momentum-temperature variables instead of using the standard conservative or primitive ones. By utilizing these new variables and the associated non-conservative form, along with adopting suitable Gaussian quadrature rules in the discretization of conservative form, we prove that this new class of schemes is well-balanced for both "lake at rest" and isobaric steady states. We validate the performance of the proposed well-balanced PAMPA method through a series of numerical experiments, demonstrating their high-order accuracy, well-balancedness, and robustness.

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From incompressible to weakly compressible Euler flows

Gert Lube *

There is some recent progress in the numerical analysis of turbulent incompressible Euler flows, see [1] which is currently restricted to low-order stabilized discontinuous Galerkin methods (dGM). Nevertheless, high-order dGM have been considered, e.g., for the 3D Taylor-Green vortex, see [2]. It seems necessary to extend the analysis to higher-order methods. Finally, we address the proposal by [3] to extend the approach to weakly compressible Euler flows.

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Failure of the least action admissibility principle in the context of the compressible Euler equations

Valentin Pellhammer*

Over the past decade, convex integration has changed our understanding of weak solutions to the multi-dimensional barotropic compressible Euler equations. In particular, it has revealed a high degree of non-uniqueness, even among solutions satisfying classical admissibility conditions such as entropy inequalities. Many of the resulting solutions may appear non-physical, prompting an active search for additional selection criteria capable of excluding such counterintuitive behavior.

Chiodaroli, De Lellis, and Kreml [1] constructed infinitely many admissible weak solutions for certain Riemann-type initial data using the method of convex integration established by De Lellis and Székelyhidi [2]. Notably, these solutions are genuinely multi-dimensional, depending on both spatial variables, despite the initial data being one-dimensional. This spontaneous symmetry breaking has intensified the debate on which solutions are physically meaningful.

One recent and notable proposal is the least action admissibility principle, introduced by H. Gimperlein, M. Grinfeld, R. J. Knops, and M. Slemrod in [3], which advocates for selecting solutions that minimize an associated action functional. This principle, inspired by variational mechanics, has been put forward as a candidate to restore uniqueness and physical relevance to weak solutions.

In this talk, we assess the effectiveness of the least action approach by applying it to a representative example, [4]. Our analysis shows that the principle discards a solution that may be regarded as physically correct. This challenges the central motivation behind the principle and suggests that either our understanding of a single physically relevant weak solution needs to be reconsidered, or the principle itself must be discarded as a viable selection criterion.

Acknowledgements

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On the dynamics of the 1D stochastic Quantum-Navier-Stokes equations

Lorenzo Pescatore*

In this talk, we consider a stochastically forced compressible viscous fluid with capillarity on the one-dimensional flat torus. The dynamics is governed by the following set of equations:

$$\begin{cases} \mathrm{d}\rho + \partial_x(\rho u)\mathrm{d}t = 0 \\ \mathrm{d}(\rho u) + \big[\partial_x(\rho u^2) + \partial_x p(\rho)\big]\mathrm{d}t = \big[\partial_x(\mu(\rho)\partial_x u) + \rho\partial_x\Big(\frac{\partial_{xx}\sqrt{\rho}}{\sqrt{\rho}}\Big)\big]\mathrm{d}t + \mathbb{G}(\rho,\rho u)\mathrm{d}W \end{cases} \tag{1}$$

$$p(\rho) = \rho^\gamma, \quad \gamma > 1, \qquad \mu(\rho) = \rho^\alpha, \quad \alpha \geq 0,$$

endowed with random initial data and periodic boundary conditions. The process W is a cylindrical Wiener process and the related stochastic integral has to be understood in the Itô sense. We prove well-posedness results for system (1) in the framework of strong and weak solutions in both PDEs and Probability sense. The analysis is performed for a wide class of density dependent viscosity coefficients $\mu(\rho) = \rho^\alpha$, $\alpha \geq 0$. In particular, we first prove a general local existence result in the class of strong pathwise solutions. Then, by means of Bresch-Desjardins entropy, higher order regularity estimates and a continuation argument we prove that for $0 \leq \alpha \leq 1/2$ the density never vanishes and thus that local strong solutions are indeed global. In the range of the viscosity parameter $1/2 < \alpha \leq 1$, vacuum regions may appear and thus only existence of weak solutions is expected. In particular, we prove the existence of weak dissipative martingale solutions by using an approximating system which provides extra dissipation properties and the convergence is based on an appropriate truncation of the velocity field in the momentum equation and a stochastic compactness argument. As a byproduct, our analysis covers also the deterministic case. This is a joint work with Donatella Donatelli and Stefano Spirito (University of L'Aquila).

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Exact solution of the Riemann problem of gas-dynamics under non-convex equation of state

G. Sirianni [*], B. Re [†], G. Guardone [‡]

The Riemann problem of the Euler equations of gas-dynamics is a cornerstone of many numerical schemes, and an important tool in the study of fundamental phenomena. Its solution under the assumption of a convex equation of state has been known for decades, but the complex wave structures arising from the loss of convexity, such as rarefaction shocks or composite fan-shock-fans [1], have given birth to a new field termed non-classical gas-dynamics. Most work in this field has been dedicated to experimentally proving the existence of these waves, currently without success in single phase conditions, while only few authors have dedicated their time advancing towards an exact solution of the non-classical Riemann problem [2].

The goal of this work is to outline the mathematics behind non-classical gas-dynamical effects, and present a novel approach which allows us to compute exact solutions of the non-classical Riemann problem of single-phase gas-dynamics under non-convex equations of state. We will present the theory and the method, showcase interesting exact solutions for pure single-component fluids such as in Fig. 1, but also equilibrium binary mixtures by employing the state-of-the-art fluid thermodynamic property library REFPROP [3]. The presented exact Riemann solver can be used to study fundamental phenomena, such as the effect of impurities on non-classical gas-dynamical waves as shown in Fig. 2, or as a powerful tool for the verification and validation of numerical solvers when employing complex equations of state or if equilibrium mixtures are considered.

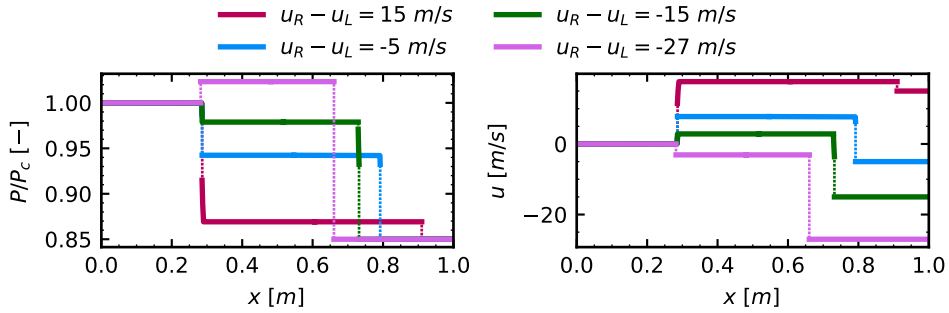

Figure 1: Exact solutions of non-classical Riemann problems for pure Dodecamethylcyclohexasiloxane (D6) at varying velocity difference $u_R - u_L$. Dotted lines represent discontinuities. At $u_R - u_L = 15\ m/s$ the solution is composed of a left running rarefaction fan-shock-fan and a right running compression shock. At $u_R - u_L = -5\ m/s$ the solution is composed of a left running rarefaction fan-shock and a right running compression shock-fan. At $u_R - u_L = -15\ m/s$ the solution is composed of a left running rarefaction fan and a right running compression shock-fan-shock. At $u_R - u_L = -27\ m/s$ the solution is composed of a left running compression shock and a right running compression shock.

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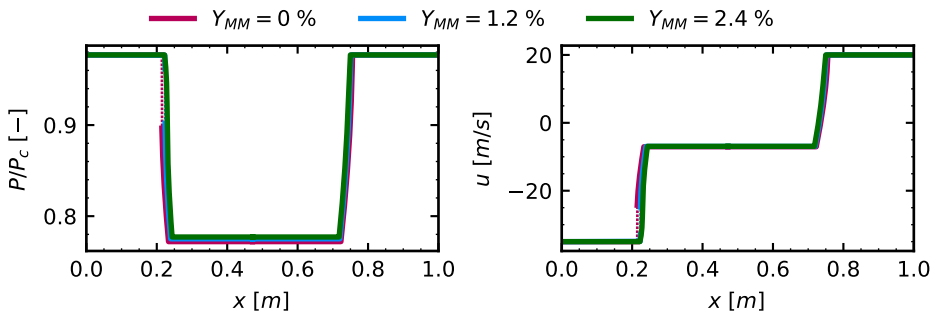

Figure 2: Non-classical Riemann problem exact solutions for a binary mixture of almost pure Tetradecamethylhexasiloxane (MD4M) with increasing amounts of Hexamethyldisiloxane (MM). Dotted lines represent discontinuities. As the mass fraction Y of impurity MM increases, the non-classical effects disappear.

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On Embedded Trefftz discontinuous Galerkin methods

Philip L. Lederer*, Christoph Lehrenfeld†, Paul Stocker‡ and Igor Voulis§

The central idea of Trefftz discontinuous Galerkin (DG) methods is to construct optimal dis- cretization spaces that minimize the number of unknowns in a system while retaining optimal approximation properties. This method has been used in the past to significantly reduce the computational cost of linear and linearized partial differential equations (PDEs).

The method is based on splitting the solution space into two parts: a local part which satisfies the local PDE and a global part which satisfies the boundary conditions and a global weak formulation of the PDE. We discuss how the local formulation can be stated to apply this method to a variety of PDEs, significantly extending the applicability of the method.

The framework we present allows us to overcome the limitations of the classical Trefftz DG method, which imposes strong restrictions on the choice of differential operators. Moreover the frameworks allows for a comprehensive error analysis and provides isights into connections to other methods. A preprint is available [1].

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Mathematical Justification of a Baer–Nunziato Model for a Compressible Viscous Fluid with Phase Transition

Florian Wendt *

In this lecture we justify a Baer–Nunziato system ([2]) including appropriate closure terms as the macroscopic description of a compressible viscous fluid that can occur in a liquid or a vapor phase in the isothermal framework. As a mathematical model for the two-phase fluid on the detailed scale we chose a non-local version of the Navier–Stokes–Korteweg equations in the one-dimensional and periodic setting. Our justification relies on anticipating the macroscopic description of the two-phase fluid as the limit system for a sequence of solutions with initial densities highly oscillating between the liquid and the vapor density. This asks for a new well-posedness result of global-in-time strong solutions for the Cauchy problem to the non-local Navier–Stokes–Korteweg system. Then we take a sequence of initial data with highly oscillating initial densities and construct for these sequence the corresponding global-in-time strong solutions. Interpreting the density as a parametrizes measure, we then extract a closed limit system consisting of a kinetic equation for the parametrized measure and a momentum equation for the velocity. This system can then be viewed as the macroscopic description of the two-phase fluid. However, expecting that the density sequence is oscillating between the liquid and the vapor density, it is reasonable to assume that the initial density distributions converge in the limit to a convex combination of two Dirac-measures. Under this assumption, we show by that this specific structure propagates in time under the kinetic equation, and, that the limit system reduces to a closed Baer–Nunziato system. With these results we extend existing results concerning the justification of Baer–Nunziato models as the macroscopic description of multifluid models ([1]) in the sense that we allow for phase transition effects on the detailed scale. The content of this lecture is based on the results obtained in [3].

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Posters

Active Flux Methods on a Sphere
Nikita Afanasev

A projection-based stabilized method for Continuous-Galerkin to solve the Euler equations
Antonio Blanco-Casares

Compressible Euler equations with transport noise
Richard Boadi

On a GRP solver for a hyperbolic model governing two-layer thin film flow
Rahul Barthwal

On the interpolation error in finite-volume methods for evolutionary PDEs on dynamic grids
Riccardo Guglielmi

A Lax-Wendroff Theorem for the Nonlinear Modified Patankar Schemes
Thomas Izgin

Optimization-based construction of multidimensional function space summation-by-parts operators
Joshua Lampert

A Structure Preserving Finite Volume Scheme for the Navier-Stokes-Korteweg Equations
Robert Sauerborn

Heat transport in rotating Rayleigh-Benard convection
Roland Welter

An asymptotic-preserving and exactly mass-conservative semi-implicit scheme for weakly compressible flows based on compatible finite elements
Enrico Zampa

	Monday, 23.06.	Tuesday, 24.06.	Wednesday, 25.06.	Thursday, 26.06.	Friday, 27.06.
08:30 - 09:00		Eduard Feireisl	Rémi Abgrall		Possibility to meet with cooperations partners
09:00 - 09:30	Introduction				
09:30 - 10:00	Eduard Feireisl			Barbara Re	
10:00 - 10:30		Coffee Break	Coffee Break		
10:30 - 11:00		Emil Wiedemann	Benjamin Gess	Coffee Break	
11:00 - 11:30	Coffee Break			Eduard Feireisl (+ closing words)	
11:30 - 12:00	Mária Lukácova-Medvidová	Dmitri Kuzmin	Piotr Mucha		
12:00 - 12:30					
12:30 - 13:00	Lunch Break	Lunch Break	Lunch Break	Lunch Break	
13:00 - 13:30					
13:30 - 14:00					
14:00 - 14:30	Rémi Abgrall	Rémi Abgrall	Excursion - Hike	Possibility to meet with cooperations partners	
14:30 - 15:00					
15:00 - 15:30					
15:30 - 16:00	Coffee Break	Coffee Break			
16:00 - 16:30	Contributed Talks	Contributed Talks			
16:30 - 17:00					
17:00 - 17:30					
17:30 - 18:00					
18:00 - 18:30	Poster Session		Conference Dinner		
18:30 - 19:00					
19:00 - 19:30					
19:30 - 20:00					
20:00 - 20:30					Mini Courses
20:30 - 21:00					
21:00 - 21:30					Invited Speakers
21:30 - 22:00					

Contributed Talks	Monday	Tuesday
	Florian Wendt	Yongle Liu
	Valentin Pellhammer	Beatrice Battisiti
	Jens Keim	Igor Voulis
	Lorenzo Pescatore	Gert Lube
	Qian Huang	Fabio Leotta
	Stephan Gerster	Giuseppe Sirianni