Compressible fluids interacting with elastic shells

Dominic Breit & Sebastian Schwarzacher

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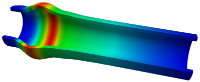
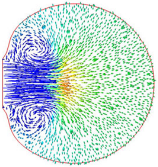
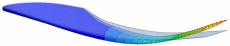
Fluid structure interaction

In this talk we will consider a compressible heat-conducting fluid which is floating in a flexible body.

- The fluid forces are interacting with a membrane that is assumed to be a part of the boundary.
- The geometry changes in time.

Examples :

Blood vessels :  Gas balloon : 



Airplane wing :

The Setting

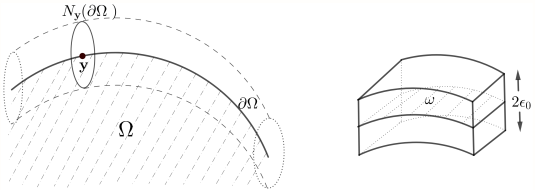


FIGURE – Left : A tubular neighbourhood of shell $\partial\Omega$ represented by bended cylinder. Right : A tiny section of the shell $\partial\Omega$ with thickness $2\varepsilon_0 > 0$.

- Abuse of notation : identify points $y \in \partial\Omega$, with points $y \in \omega$, for simplicity assume ω flat torus ;
- Ω can be parametrised by $\varphi : \omega \to \mathbb{R}^3$;
- For $\eta : \omega \to \mathbb{R}$ consider $\varphi : \omega \to \mathbb{R}^3$ given by

$$\varphi_\eta(y) = \varphi(y) + \eta(y)\nu(y), \quad y \in \omega.$$

Koiter's elastic energy

Elastic energy of the deformation given by

$$K(\eta) = \frac{1}{2}\varepsilon_0 \int_\omega \mathbb{C} : \mathbb{G}(\eta) \otimes \mathbb{G}(\eta)\, \mathrm{d}y + \frac{1}{6}\varepsilon_0^3 \int_\omega \mathbb{C} : \mathbb{R}^\sharp(\eta) \otimes \mathbb{R}^\sharp(\eta)\, \mathrm{d}y$$

- $\mathbb{G}$ contains covariant components of change of metric tensor ;
- $\mathbb{R}^\sharp$ contains covariant components of change of curvature tensor ;
- $\mathbb{C} = (C^{ijkl})^2_{i,j,k,l=1}$ contains contravariant components of the shell elasticity ;
- K is coercive on $W^{2,2}(\omega)$ but not continuous : most critical term in $K(\eta)$ behaves as $\int_\omega |\nabla^2\eta|^2 |\nabla\eta|^4\, \mathrm{d}y$.

The PDE in the interior

For $\eta : I \times \omega \to \mathbb{R}$ given, denote $\Omega_{\eta(t)}$ the variable in time domain, by $I \times \Omega_\eta = \bigcup_{t\in I} \{t\} \times \Omega_{\eta(t)}$ the deformed time-space cylinder :

$$\partial\Omega_{\eta(t)} = \{\varphi(y) + \eta(t,y)\nu(y) : y \in \omega\}.$$

The fluid is heat-conducting, compressible and viscous

$$\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, && \text{in } I \times \Omega_\eta,\\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) &= \operatorname{div} \mathbf{S}(\vartheta, \nabla \mathbf{u}) - \nabla p + \varrho \mathbf{f} && \text{in } I \times \Omega_\eta,\\
\partial_t(\varrho e) + \operatorname{div}(\varrho e \mathbf{u}) &= \mathbf{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - p \operatorname{div} \mathbf{u} &&\\
&\quad - \operatorname{div} \mathbf{q}(\vartheta, \nabla \vartheta) + \varrho H && \text{in } I \times \Omega_\eta,\\
\mathbf{u}(t, x + \boldsymbol{\eta}(x)) &= \partial_t \boldsymbol{\eta}(t,x) && \text{in } I \times \omega.
\end{aligned}$$

The shell is driven by Koiter-energy $\rightsquigarrow$ equation for the shell is

$$\begin{aligned}
\partial_t^2 \eta + K'(\eta) &= g + \nu \cdot \big(-\boldsymbol{\tau} \nu_\eta \big) \circ \varphi_{\eta(t)} \left| \det D\varphi_{\eta(t)} \right| \quad \text{in } I \times \omega,\\
\boldsymbol{\tau} &= \mathbf{S}(\vartheta, \nabla \mathbf{u}) - p\mathcal{I}.
\end{aligned}$$

Constitutive relations

- Newton's rheological law (with $\lambda, \mu \cong 1 + \vartheta$)

$$\mathbf{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\left(\frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2} - \frac{1}{3} \operatorname{div} \mathbf{u} \mathcal{I}\right) + \lambda(\vartheta) \operatorname{div} \mathbf{u} \mathcal{I};$$

- Heat flux determined by Fourier's law (with $\kappa \cong 1 + \vartheta^3$)

$$\mathbf{q}(\vartheta, \nabla \vartheta) = -\varkappa(\vartheta)\nabla\vartheta = -\nabla\mathcal{K}(\vartheta), \quad \mathcal{K}(\vartheta) = \int_0^\vartheta \varkappa(z)\,\mathrm{d}z;$$

- p and e related to (specific) entropy s through Gibbs' equation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right) \quad \text{for all} \quad \varrho, \vartheta > 0,$$

$$p(\varrho, \vartheta) = \varrho^\gamma + \varrho\vartheta + \frac{a}{3}\vartheta^4, \; s(\varrho, \vartheta) = \log(\vartheta^{c_v}) - \log \varrho + \frac{4a}{3}\frac{\vartheta^3}{\varrho}.$$

Entropy/energy balance balance

The Second law of thermodynamics is enforced through

the entropy balance equation

$$\partial_t(\varrho s(\varrho,\vartheta)) + \operatorname{div}(\varrho s(\varrho,\vartheta)\mathbf{u}) = -\operatorname{div}\left(\frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\right) + \sigma + \varrho\frac{H}{\vartheta},$$

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\right).$$

- Approach based on entropy balance by Feireisl-Novotný ('09), earlier approach based on energy balance by Feireisl ('04).
- Energy conserved in the entropy approach, here

$$\mathcal{E} = \int_{\Omega_\eta}\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)\mathrm{d}x + \int_\omega \frac{|\partial_t\eta|^2}{2}\,\mathrm{d}y + K(\eta).$$

Weak formulation (1)

Let $\eta : I \times \omega \to \mathbb{R}$ and consider a coordinate map $\mathbf{\Psi}_\eta : \Omega \to \Omega_\eta$.

Reynolds transport theorem :

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\eta(t)}} g \, \mathrm{d}x = \int_{\Omega_{\eta(t)}} \partial_t g \, \mathrm{d}x + \int_{\partial\Omega_{\eta(t)}} \partial_t \eta \circ \boldsymbol{\varphi}^{-1} \circ \mathbf{\Psi}_\eta^{-1} \nu \cdot \nu_\eta g \, \mathrm{d}\mathcal{H}^2,$$

The weak continuity equation : integration by parts implies for $\psi \in C^\infty(I \times \overline{\Omega})$

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho\psi \, \mathrm{d}x \, \mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi \Big) \, \mathrm{d}x \, \mathrm{d}t = 0,$$

if $\mathbf{u} \circ \mathbf{\Psi}_\eta \circ \boldsymbol{\varphi} = \partial_t \eta \nu$ on $\partial\Omega_{\eta(t)}$.

Weak formulation (2)

The weak momentum equation :
For $(b, \varphi) \in C^\infty(\omega) \times C^\infty(\bar{I} \times \mathbb{R}^3)$ with $\mathrm{tr}_\eta \varphi = b\nu$

$$\int_I \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \varphi \,\mathrm{d}x - \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \varphi \,\mathrm{d}x \right) \mathrm{d}t$$
$$+ \int_I \int_{\Omega_\eta} \Big(\mathbf{S}(\vartheta, \nabla \mathbf{u}) : \nabla \varphi - p(\vartheta, \vartheta) \,\mathrm{div}\, \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t$$
$$+ \int_I \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \partial_t \eta b \,\mathrm{d}y - \int_\omega \partial_t \eta \, \partial_t b \,\mathrm{d}y + \int_\omega K'(\eta)\, b \,\mathrm{d}y \right) \mathrm{d}t$$
$$= \int_I \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \varphi \,\mathrm{d}x \,\mathrm{d}t + \int_I \int_\omega g \, b \,\mathrm{d}y \,\mathrm{d}t.$$

Weak formulation (3)

The weak entropy balance : We have

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho s(\varrho,\vartheta)\,\psi - \int_I \int_{\Omega_\eta} \left(\varrho s(\varrho,\vartheta)\partial_t \psi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\right)$$
$$\geq \int_I \int_{\Omega_\eta} \frac{1}{\vartheta}\Big(\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u} + \frac{\varkappa(\vartheta)}{\vartheta}|\nabla\vartheta|^2\Big)\psi$$
$$- \int_I \int_{\Omega_\eta} \frac{\varkappa(\vartheta)\nabla\vartheta}{\vartheta}\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t + \int_I \int_{\Omega_\eta} \frac{\varrho}{\vartheta} H\psi$$

for all $\psi \in C^\infty(\bar{I}\times\mathbb{R}^3)$ with $\psi \geq 0$. Moreover, we have $\lim_{r\to 0} \varrho s(\varrho,\vartheta)(t) \geq \varrho_0 s(\varrho_0,\vartheta_0)$ and $\partial_{\nu_\eta}\vartheta|_{\partial\Omega_\eta} \leq 0$.

Renormalized continuity equation

The renormalized continuity equation :

For $\psi \in C^\infty(I \times \overline{\Omega})$ and $\theta \in C^1(\mathbb{R}^+)$ with $\theta(0) = 0$

$$0 = \int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \theta(\varrho)\, \psi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\theta(\varrho)\partial_t \psi + \theta(\varrho)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_I \int_{\Omega_\eta} (\varrho\theta'(\varrho) - \theta(\varrho))\,\mathrm{div}\,\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t.$$

- Major ingredient for compressible Navier–Stokes equations.
- Introduced by Di Perna-Lions '89.
- Version above : Breit-Schwarzacher, '18.

Main theorem

Theorem (Breit-Schwarzacher, Ann. SNS Pisa, to appear)

Let $\gamma > \frac{12}{7}$ ($\gamma > 1$ in two dimensions). Under natural assumptions on the data there exists a weak solution $(\eta, \mathbf{u}, \varrho, \vartheta)$ with satisfies the energy balance

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_{\Omega_\eta} \varrho H \,\mathrm{d}x + \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \mathbf{u} \,\mathrm{d}x + \int_\omega g\, \partial_t \eta \,\mathrm{d}y,$$

$$\mathcal{E} = \int_{\Omega_\eta} \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho\theta\Big) \,\mathrm{d}x + \int_\omega \frac{|\partial_t \eta|^2}{2} \,\mathrm{d}y + K(\eta).$$

The interval of existence is of the form $I = (0, t)$, $t < T$ only if

- $\Omega_{\eta(s)}$ approaches a self-intersection when $s \to t$ or
- the Koiter energy degenerates (namely, if $\lim_{s\to t} \overline{\gamma}(s, y) = 0$ for some point $y \in \omega$).

Main theorem

Theorem (Breit, Schwarzacher, ARMA '18)

Let $\gamma > \frac{12}{7}$ ($\gamma > 1$ in two dimensions). There is a weak solution $(\eta, \mathbf{u}, \varrho)$. The solution satisfies the energy estimate

$$\sup_{t\in I}\int_{\Omega_\eta}\varrho|\mathbf{u}|^2\,\mathrm{d}x+\sup_{t\in I}\int_{\Omega_\eta}\varrho^\gamma\,\mathrm{d}x+\int_I\int_{\Omega_\eta}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\sup_{t\in I}\int_\omega|\partial_t\eta|^2\,\mathrm{d}y+\sup_{t\in I}\int_\omega|\nabla^2\eta|^2\leq c(\mathbf{q}_0,\varrho_0,\mathbf{f},g,\eta_0,\eta_1).$$

1. Incompressible analogue : Lengeler & Růžička, (ARMA, '14).
2. Restriction to linear shell models !

Sequential compactness

Assume there is a sequence of solutions $(\eta_n, \mathbf{u}_n, \varrho_n)$ which enjoys suitable regularity properties and satisfies the energy estimate uniformly in n.

How can we pass to the limit in the equation ?

- passage to the limit in the convective terms $\varrho_n \mathbf{u}_n$ and $\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n$ by local arguments and global integrability.
  No problems with the moving boundary !
  Much easier than incompressible case !
- Main problem (typical for compressible Navier–Stokes) :
  Passing to the limit in the nonlinear pressure $p(\varrho_n) = \varrho_n^\gamma$.

Higher integrability of the pressure (1)

- A priori $p(\varrho_n)$ only bounded in $L^\infty(L^1)$
  $\Rightarrow$ Concentrations possible !
- Improve integrability by "computing the pressure" : Use globally Bogovskiĭ-operator $\approx \mathrm{div}^{-1}$ or locally $\Delta^{-1}\mathrm{div}$:

$$\int p(\varrho_n)\varrho_n = \int \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla^2\Delta^{-1}\varrho_n + \dots \quad (\text{if } \gamma > 3),$$

$$\int p(\varrho_n)\varrho_n^\Theta = \int \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla^2\Delta^{-1}\varrho_n^\Theta + \dots \quad (\text{if } \Theta \leq \frac{3}{2}\gamma - 1);$$

- Bogovskiĭ-operator requires Lipschitz boundary $\Rightarrow$ standard approach only gives higher integrability locally.
- How to exclude concentrations at the boundary ?

Higher integrability of the pressure (2)

Lemma

There is a measurable set $A_\kappa \Subset I \times \Omega_{\eta_n}$ such that for all $n \geq n_0$

$$\int_{I\times\mathbb{R}^3\setminus A_\kappa} a\varrho_n^\gamma \chi_{\Omega_{\eta_n}} \,\mathrm{d}x\,\mathrm{d}t \leq \kappa.$$

Construct test-function φ_n such that :

- zero boundary conditions on $\partial\Omega_{\eta_n}$;
- $\operatorname{div}\varphi_n \geq K_\kappa$ close to the boundary ;
- Critical term $\int_I \int_{\Omega_{\eta_n}} \varrho_n \mathbf{u}_n \,\partial_t \varphi_n \,\mathrm{d}x\,\mathrm{d}t$ with $\partial_t \varphi_n \sim \partial_t \eta_n = \mathbf{u}_n \circ \boldsymbol{\Psi}_n|_{\partial\Omega_{\eta_n}} \in L^2(L^{4^-})$;
- Since $\varrho_n \mathbf{u}_n \in L^2(L^{\frac{6\gamma}{\gamma+6}})$ we need $\gamma > \frac{12}{7}$.

Four layer approximation scheme

- Artificial pressure (δ-layer) : replace $p(\varrho) = a\varrho^\gamma$ by $p_\delta(\varrho) = a\varrho^\gamma + \delta\varrho^\beta$ where β is chosen large enough.
- Artificial viscosity (ε-layer) : add $\varepsilon\Delta\varrho$ to the right-hand side of the continuity equation.
- Regularization of the boundary (κ-layer) : Replace the underlying domain Ω_η by Ω_{η_κ} where η_κ is a suitable regularization of η. Accordingly, the convective terms and the pressure have to be regularized as well.
- Finite-dimensional approximation (N-layer) : the momentum equation has to be solved by means of a Galerkin-approximation.

The first two layers are common in the theory of compressible Navier–Stokes equations, see Feireisl et al. Third layer is needed due to the low regularity of η.

Strong convergence of the temperature

The entropy balance

$$\partial_t(\varrho s(\varrho,\vartheta)) + \operatorname{div}(\varrho s(\varrho,\vartheta)\mathbf{u}) \geq -\operatorname{div}\left(\frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\right) + \sigma + \varrho\frac{H}{\vartheta},$$
$$\sigma = \frac{1}{\vartheta}\left(\mathbf{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\right).$$

- Gives control of $\nabla\vartheta$ using $\mathbf{q}(\vartheta,\nabla\vartheta) = -\varkappa(\vartheta)\nabla\vartheta$;
- No control of $\partial_t\vartheta_n$ (recall $\varrho s(\varrho,\vartheta) = \frac{4a}{3}\vartheta^3$)
- Use monotonicity of s combined with div-curl-lemma in space-time (Feireils-Novotný) ;
- Local argument $\rightsquigarrow$ no problems with moving domain.

Total energy

$$\mathcal{E} = \int_{\Omega_\eta} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \mathrm{d}x + \int_\omega \frac{|\partial_t \eta|^2}{2} \, \mathrm{d}y + K(\eta).$$

- Terms related to the fluid do not cause problems ;
- Coercivity of K gives control of $\nabla^2 \eta$, not enough to pass to the limit (not even in linear case) ;
- Fractional derivative of $\partial_t \eta$ only in space (trace theorem) ;
- Use information from coupled the momentum equation to get compactness of $\partial_t \eta$ by uniform continuity in time (approach by Lengeler-Růžička or abstract framework by Muha-Schwarzacher).

Fractional difference quotient for some $s > 0$

$$\int_I \|\Delta_h^s \nabla^2 \eta\|_{L^2(\omega)}^2 \,\mathrm{d}t \leq c \;\Rightarrow\; \int_I \|\eta\|_{W^{2+s,2}(\omega)}^2 \,\mathrm{d}t \leq c.$$

- Muha-Schwarzacher : use test-function $(\boldsymbol{\varphi}, \varphi)$ with

$$\varphi = \Delta_{-h}^s \Delta_h^s \eta - \mathscr{K}_\eta(\Delta_{-h}^s \Delta_h^s \eta),$$
$$\boldsymbol{\varphi} = \big(\mathscr{F}_\eta^{\mathrm{div}}(\Delta_{-h}^s \Delta_h^s \eta - \mathscr{K}_\eta(\Delta_{-h}^s \Delta_h^s \eta));$$

- Pressure only in L^1-globally !
- Most critical term is (requires $\gamma > \frac{12}{7}$)

$$\int_I \int_{\Omega_{\eta(t)}} \varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}t.$$

Three layer approximation scheme

- Artificial pressure (δ-layer) : replace $p(\varrho) = \varrho^\gamma + a\vartheta^4$ by $p_\delta(\varrho) = \varrho^\gamma + \delta\varrho^\beta + a\vartheta^4$ where β is chosen large enough ;
- Artificial viscosity (ε-layer) : add $\varepsilon\Delta\varrho$ to the right-hand side of the continuity equation ;
- Galerkin-approximation (N-layer) $\rightsquigarrow$ fixed point for linearised problem on the Galerkin level (use basis functions $\mathbf{\Psi}_\eta^{-1} \circ \tilde{\boldsymbol{\omega}}_k$) ;

How to prove strict positivity of the temperature ?

- Internal energy equation continuous in space, velocity and shell finite dimensional ;
- Transform to reference domain ($\det \nabla \mathbf{\Psi}_\eta \neq 1$) ;
- Regularity theory and minimum principle for transformed equation.