Traces of functions of bounded $\mathbb{A}$-variation and variational problems with linear growth

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28.06.2017

Linear growth functionals (1)

Find a minimizer $\mathbf{u} : \Omega \to \mathbb{R}^N$, $\Omega \subset \mathbb{R}^n$ bounded and Lipschitz,

of the variational problem

$$\mathfrak{F}[\mathbf{u}] = \int_\Omega f(x, D\mathbf{u})\,\mathrm{d}x, \quad \mathbf{u}|_{\partial\Omega} = \mathbf{u}_0.$$

- $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ has linear growth and is quasi-convex, i.e.

$$c_0|\mathbf{Z}| - C_0 \leq f(x, \mathbf{Z}) \leq c_1|\mathbf{Z}| + C_1,$$
$$f(x, \mathbf{A}) \leq \int_{(0,1)^n} f(y, \mathbf{A} + D\varphi)\,\mathrm{d}y \quad \forall \varphi \in C_0^\infty((0,1)^n);$$

- $\mathfrak{F}$ defined on $W^{1,1}(\Omega)$, minimization in $\mathbf{u}_0 + W_0^{1,1}(\Omega)$ fails.

Linear growth functionals (2)

Extended functional for $\mathbf{u} \in \mathrm{BV}(\Omega)$

$$\overline{\mathfrak{F}}_{\mathbf{u}_0}[\mathbf{u}] := \int_\Omega f\left(x, \frac{\mathrm{d}D\mathbf{u}}{\mathrm{d}\mathscr{L}^n}\right) \mathrm{d}\mathscr{L}^n + \int_\Omega f^\infty\left(x, \frac{\mathrm{d}D\mathbf{u}}{\mathrm{d}|D^s\mathbf{u}|}\right) \mathrm{d}|D^s\mathbf{u}| + \int_{\partial\Omega} f^\infty\Big(x, \nu_{\partial\Omega} \otimes \mathrm{tr}(\mathbf{u} - \mathbf{u}_0)\Big) \mathrm{d}\mathcal{H}^{n-1}.$$

- $f^\infty : \overline{\Omega} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is the strong recession function, i.e.

$$f^\infty(x, \mathbf{A}) := \lim_{\substack{x' \to x \\ \mathbf{A}' \to \mathbf{A} \\ t \to \infty}} \frac{f(x', t\mathbf{A}')}{t};$$

- infima coincide: $\inf_{\mathbf{u} \in \mathbf{u}_0 + W_0^{1,1}(\Omega)} \mathfrak{F}[\mathbf{u}] = \min_{\mathbf{u} \in \mathrm{BV}(\Omega)} \overline{\mathfrak{F}}_{\mathbf{u}_0}[\mathbf{u}]$.

Linear growth functionals (3)

- Existence of a minimizer by the direct method in the calculus of variations.
- Ambrosio-Dal Maso (1992), Fonseca-Müller (1993): $\overline{\mathfrak{F}}_{\mathbf{u}_0}$ is lower semi-continuous on $\mathrm{BV}(\Omega)$ w.r.t. weak*-topology! Proof uses blow-up method and Alberti's rank-one theorem (1993).
- There is a linear continuous operator $\mathrm{tr}: \mathrm{BV}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ s.t. $\mathrm{tr}\,\mathbf{u} = \mathbf{u}|_{\partial\Omega}$ for $\mathbf{u} \in C^0(\overline{\Omega})$.
- Continuity w.r.t. the strict topology: $\mathbf{u}_n \to^s \mathbf{u}$ iff $\mathbf{u}_n \to \mathbf{u}$ in $L^1(\Omega)$ and $|D\mathbf{u}_n|(\Omega) \to |D\mathbf{u}|(\Omega)$.

The trace operator on $W^{1,1}$

Gagliardo (1957): There is a surjective linear continuous operator

$$\mathrm{tr} : W^{1,1}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1}), \quad \mathrm{tr}\, u = u|_{\partial\Omega} \quad \forall u \in C^0(\overline{\Omega}).$$

- Extends to $\mathrm{BV}(\Omega)$ by smooth approximation (strict topology): Anzellotti-Giaquinta (1978).
- Proof by fundamental theorem of calculus: for $u \in C_c^\infty(\mathbb{R}^n)$

$$u(x_1, \ldots, x_{n-1}, 0) = \int_{-\infty}^0 \partial_n u(x_1, \ldots, x_{n-1}, t)\,\mathrm{d}t,$$
$$\Rightarrow \|u(\cdot, 0)\|_{L^1(\mathbb{R}^{n-1})} \leq \|Du\|_{L^1(\mathbb{R}^n)}.$$

The space $\mathrm{BD}(\Omega)$

If $N = n$ let $\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ be the symmetric gradient

$$\mathrm{LD}(\Omega) := \{\mathbf{u} \in L^1(\Omega) : \mathcal{E}(\mathbf{u}) \in L^1(\Omega)\},$$
$$\mathrm{BD}(\Omega) := \{\mathbf{u} \in L^1(\Omega) : \mathcal{E}(\mathbf{u}) \in \mathcal{M}(\Omega)\}.$$

- Introduced by Suquet (1978), Matthies-Strang-Christiansen (1979), Temam-Strang (1980);
- Proper superspace of $\mathrm{BV}(\Omega)$ (Ornstein's non-inequality in L^1);
- Study linear-growth functionals

$$\mathfrak{F}[\mathbf{u}] = \int_\Omega f(x, \mathcal{E}(\mathbf{u}))\,\mathrm{d}x, \quad \mathbf{u}|_{\partial\Omega} = \mathbf{u}_0.$$

Linear growth functionals on BD(Ω)

Extended functional for $\mathbf{u} \in \mathrm{BD}(\Omega)$

$$\overline{\mathfrak{F}}_{\mathbf{u}_0}[\mathbf{u}] := \int_\Omega f\left(x, \frac{\mathrm{d}\mathcal{E}(\mathbf{u})}{\mathrm{d}\mathscr{L}^n}\right)\mathrm{d}\mathscr{L}^n + \int_\Omega f^\infty\left(x, \frac{\mathrm{d}\mathcal{E}(\mathbf{u})}{\mathrm{d}|\mathcal{E}^s(\mathbf{u})|}\right)\mathrm{d}|\mathcal{E}^s(\mathbf{u})| + \int_{\partial\Omega} f^\infty\Big(x, \nu_{\partial\Omega} \otimes_{sym} \mathrm{tr}(\mathbf{u} - \mathbf{u}_0)\Big)\,\mathrm{d}\mathcal{H}^{n-1}.$$

- Rindler (2011): Lower semi-continuity via rigidity and Young measures (an analogone of Alberti's rank one theorem was not known).
- Strang-Temam (1980): There is a linear continuous operator $\mathrm{tr} : \mathrm{BD}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ s.t. $\mathrm{tr}\,\mathbf{u} = \mathbf{u}|_{\partial\Omega}$ for $\mathbf{u} \in C^0(\overline{\Omega})$. $\mathbf{u} \in \mathrm{BD}(\Omega)$ iff $\xi D(\mathbf{u}\cdot\xi) \in \mathcal{M}(\Omega)$ for all $\xi \in \mathbb{R}^n$.

Trace-free symmetric gradients

$\mathcal{E}^D(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{1}{n}\operatorname{div}(\mathbf{u})$ = trace-free symmetric gradient

$$W^{\mathcal{E}^D,1}(\Omega) := \{\mathbf{u} \in L^1(\Omega) : \mathcal{E}^D(\mathbf{u}) \in L^1(\Omega)\},$$
$$\mathrm{BV}^{\mathcal{E}^D}(\Omega) := \{\mathbf{u} \in L^1(\Omega) : \mathcal{E}^D(\mathbf{u}) \in \mathcal{M}(\Omega)\}.$$

- If $n \geq 3$ $N(\mathcal{E}^D)$ = killing vectors (quadratic polynomials);
- If $n = 2$ $N(\mathcal{E}^D)$ = holomorphic functions;
- Fuchs- Repin (2010): no trace if $n = 2$, consider $B_1 \ni z \mapsto (z-1)^{-1} \in \mathbb{C}$;
- What happens if $n \geq 3$? No control of $\partial_i u^i$ or $\operatorname{div} \mathbf{u}$!

General differential operators

Linear maps $\mathbb{A}_\alpha : \mathbb{R}^N \to \mathbb{R}^K$ (e.g. $K = N \times n$) s.t.

$$\mathbb{A} = \sum_{\alpha=1}^{N} \mathbb{A}_\alpha \partial_\alpha$$

- The *symbol mapping* $\mathbb{A}[\xi] : \mathbb{R}^N \to \mathbb{R}^K$ is defined by

$$\mathbb{A}[\xi]v := v \otimes_{\mathbb{A}} \xi := \sum_{\alpha=1}^{n} \xi_\alpha \mathbb{A}_\alpha v.$$

- $\mathbb{A}$ is $\mathbb{R}$-*elliptic* if $\mathbb{A}[\xi]\colon \mathbb{R}^N \to \mathbb{R}^K$ is injective $\forall \xi \in \mathbb{R}^n \setminus \{0\}$;
- $\mathbb{A}$ is $\mathbb{C}$-*elliptic* if $\mathbb{A}[\xi]\colon \mathbb{C}^N \to \mathbb{C}^K$ is injective $\forall \xi \in \mathbb{C}^n \setminus \{0\}$;
- $\mathbb{A}$ is $\mathbb{C}$-*elliptic* iff $\dim(N(\mathbb{A})) < \infty$.

The space $\mathrm{BV}^{\mathbb{A}}(\Omega)$

$\mathbb{A}$ linear, homogeneous, constant coefficient

$$W^{\mathbb{A},1}(\Omega) := \{\mathbf{u} \in L^1(\Omega) : \mathbb{A}\mathbf{u} \in L^1(\Omega)\},$$
$$\mathrm{BV}^{\mathbb{A}}(\Omega) := \{\mathbf{u} \in L^1(\Omega) : \mathbb{A}\mathbf{u} \in \mathcal{M}(\Omega)\}.$$

- Van Schaftingen (2013): $W^{\mathbb{A},1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n)$ iff $\mathbb{A}$ is cancelling, i.e. $\bigcap_{\xi \neq 0} \mathbb{A}[\xi](\mathbb{R}^n) = \{0\}$;
- $\mathbb{C}$-ellipticity $\Rightarrow$ cancelling;
- Classical Gagliardo-Nirenberg-Sobolev inequality if $\mathbb{A} = D$;
- Strauss-inequality if $\mathbb{A} = \mathcal{E}$ (1971).

Poincaré's inequality

$\mathbb{A}$ be $\mathbb{C}$-elliptic, B a ball,

$$\inf_{\mathbf{q}\in N(\mathbb{A})} \|\mathbf{u}-\mathbf{q}\|_{L^1(B)} \leq \|\mathbf{u}-\Pi_B\mathbf{u}\|_{L^1(B)} \leq c\,\ell(B)\,|\mathbb{A}\mathbf{u}|(B),$$

- Π_B is the L^2-orthogonal projection onto $N(\mathbb{A})$;
- Based on representation formula by Kalamajska (1994) and smooth approximation;
- Elements of $N(\mathbb{A})$ are polynomials;
- If $\mathbf{u}=0$ "somewhere" then $\|\mathbf{u}\|_{L^1(B)} \leq c\,\ell(B)\,|\mathbb{A}\mathbf{u}|(B)$.

Alberti-type theorem

De Philippis-Rindler (2016): for $\mathcal{A}$-free measure μ

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu^s|} \in \Lambda_{\mathcal{A}} := \bigcup_{\xi \neq 0} \ker(\mathcal{A}[\xi]) = \bigcup_{\xi \neq 0} \mathbb{A}[\xi](\mathbb{R}^n) \quad |\mu^s| - \text{a.e.}$$

- $\Lambda_{\mathcal{A}}$ is called characteristic wave cone;
- $\mathbb{A}$ is potential to $\mathcal{A}$ (e.g. $\mathbb{A} = D$ and $\mathcal{A} = \mathrm{curl}$);
- If $\mathbf{u} \in \mathrm{BV}(\Omega)$ then $\frac{\mathrm{d}D\mathbf{u}}{\mathrm{d}|D^s\mathbf{u}|} \in \{v \otimes \xi\}$ $|D^s\mathbf{u}|$-a.e.;
- If $\mathbf{u} \in \mathrm{BD}(\Omega)$ then $\frac{\mathrm{d}\mathcal{E}(\mathbf{u})}{\mathrm{d}|\mathcal{E}^s(\mathbf{u})|} \in \{v \otimes_{sym} \xi\}$ $|\mathcal{E}^s(\mathbf{u})|$-a.e.;
- If $\mathbf{u} \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ then $\frac{\mathrm{d}\mathbb{A}\mathbf{u}}{\mathrm{d}|\mathbb{A}^s\mathbf{u}|} \in \{v \otimes_{\mathbb{A}} \xi\}$ $|\mathbb{A}^s\mathbf{u}|$-a.e.

Linear growth functionals on $\mathrm{BV}^{\mathbb{A}}(\Omega)$

Extended functional for $\mathbf{u} \in \mathrm{BV}^{\mathbb{A}}(\Omega)$

$$\overline{\mathfrak{F}}[\mathbf{u}] := \int_{\Omega} f\left(x, \frac{\mathrm{d}\mathbb{A}\mathbf{u}}{\mathrm{d}\mathscr{L}^n}\right) \mathrm{d}\mathscr{L}^n + \int_{\Omega} f^{\infty}\left(x, \frac{\mathrm{d}\mathbb{A}\mathbf{u}}{\mathrm{d}|\mathbb{A}^s\mathbf{u}|}\right) \mathrm{d}|\mathbb{A}^s\mathbf{u}|$$

- Arroyo-Rabasa, De Philippis, Rindler, (2017): Lower semi-continuity via Alberti-type theorem of functionals $\overline{\mathfrak{F}}[\mu]$.
- Baía, Chermisi, Matias, Santos (2013): Lower semi-continuity via Young measures;
- Trace-part not included so far!

Main result

Let $\mathbb{A}$ be a $\mathbb{C}$-elliptic operator and Ω a bounded Lipschitz domain.

Breit, Diening, Gmeineder (2017): $\exists$ linear continuous operator

$$\operatorname{tr} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1}), \quad \operatorname{tr} \mathbf{u} = \mathbf{u}|_{\partial\Omega} \quad \forall u \in C^0(\overline{\Omega}).$$

- Main difficulty: estimate, even for smooth functions;
- Extension of Fuchs-Repin counterexample for $\mathcal{E}^D$ shows: $\mathbb{C}$-elliptic is also necessary;
- Lipschitz boundary can be weakened to domains satisfying
  1. Ω is an NTA (non-tangentially accessible);
  2. Ω is Ahlfors regular, i.e. there is $R > 0$ and $M > 0$ s.t.

$$\frac{1}{M} r^{n-1} \leq \mathcal{H}^{n-1}(B_r(x) \cap \partial\Omega) \leq M r^{n-1} \quad \forall r \in (0, R).$$

NTA domains

A domain $\Omega \subset \mathbb{R}^n$ is an NTA (non-tangentially accessible) domain if it satisfies the interior corkscrew condition, the exterior interior corkscrew condition and the interior Harnack chain condition.

- We say that Ω satifies the *interior corkscrew condition* if there exists $R > 0$ and $M > 2$ such that for all $x \in \partial\Omega$ and all $r \in (0, R)$ there exists a $y \in \Omega$ such that

$$\frac{1}{M} r \leq |x - y| \leq r \qquad \text{and} \qquad B(y, r/M) \subset \Omega.$$

- We say that Ω satifies the *exterior corkscrew condition* if $\mathbb{R}^n \setminus \Omega$ satisfies the interior corkscrew condition.

Interior Harnack chain condition

- We say that $\Omega \subset \mathbb{R}^n$ satisfies the *(interior) Harnack chain condition* if any interior points $y_1, y_2 \in \Omega$ can be connected via a chain of proportional balls $B_1, \ldots, B_J$ in Ω satisfying
  1. $y_1 \in B_1,\ y_2 \in B_J$,
  2. $\ell(B_j) \sim d(B_j, \partial\Omega)$ for $j = 1, \ldots, J$,
  3. $\ell(B_j) \geq c \min\left\{d(y_1, B_j), d(y_2, B_j)\right\}$ for $j = 1, \ldots, J$,
  4. J uniformly bounded in terms of $\frac{|y_1 - y_2|}{\min\{d(y_1,\partial\Omega), d(y_2,\partial\Omega)\}}$.

Covering by balls

- For each $j \in \mathbb{Z}$, let $(B_{j,k})_k$ denote a (countable) cover of balls of $\mathbb{R}^n$ with diameter $\sim 2^{-j}$;
- For each j let $(\eta_{j,k})_k$ we find a partition of unity with respect to the $(B_{j,k})_k$;
- Define the 2^{-j}-neighbourhood U_j of $\partial\Omega$ by

$$U_j := \{x \in \Omega \,:\, d(x, \partial\Omega) < 2^{-j}\}.$$

- Interior corkscrew condition $\Rightarrow$ for each ball $B_{j,k}$ close to the boundary there is a *reflected ball* $B^{\sharp}_{j,k}$ close by.
- Harnack chain condition $\Rightarrow$ connect two reflected balls of neighboring balls by a small chain of balls $W_1, \ldots, W_\gamma$ and set

$$\Omega(B^{\sharp}_{j,k}, B^{\sharp}_{l,m}) := \bigcup_{\beta=1}^{\gamma} W_\beta.$$

Ideas of the proof (1)

Use local projections $\Pi_{j,k} = \Pi_{B^\sharp_{j,k}}$ and set

$$T_j\mathbf{u} := \mathbf{u} - \rho_j \sum_k \eta_{j,k}\big(\mathbf{u} - \Pi_{j,k}\mathbf{u}\big) = (1-\rho_j)\mathbf{u} + \rho_j \sum_k \eta_{j,k}\Pi_{j,k}\mathbf{u}.$$

- ρ_j smooth s.t. $\chi_{U_{j+1}} \leq \rho_j \leq \chi_{U_j}$;
- T_j smooth at the boundary!
- If $\mathbf{u} \in \mathrm{BV}^{\mathbb{A}}(\Omega)$, then in $\mathrm{BV}^{\mathbb{A}}(\Omega)$

$$\mathbf{u} = T_{j_0}\mathbf{u} + \sum_{l=j_0}^{\infty}\big(T_{l+1}\mathbf{u} - T_l\mathbf{u}\big) = \lim_{j\to\infty} T_j\mathbf{u}.$$

Ideas of the proof (2)

Let $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. Then for some $k_0 \in \mathbb{N}$

$$\|\mathrm{tr}(T_{j+1}\mathbf{u}) - \mathrm{tr}(T_j\mathbf{u})\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}\mathbf{u}|(U_{j-k_0} \setminus U_{j+k_0}).$$

Proof: From defintion of T_j

$$\mathrm{tr}(T_{j+1}\mathbf{u}) - \mathrm{tr}(T_j\mathbf{u}) = \sum_{k,m} \mathrm{tr}\left(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}\mathbf{u} - \Pi_{j,m}\mathbf{u})\right),$$

where the sums are locally finite sums. Hence,

$$\|\mathrm{tr}(T_{j+1}\mathbf{u}) - \mathrm{tr}(T_j\mathbf{u})\|_{L^1(\partial\Omega)} \leq \sum_{k,m} \left\|\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}\mathbf{u} - \Pi_{j,m}\mathbf{u})\right\|_{L^1(\partial\Omega)}.$$

Ideas of the proof (3)

We only have to consider those k, m with $B_{j+1,k} \cap B_{j,m} \neq \emptyset$. For such k, m

$$\begin{aligned}\left\|\mathrm{tr}\left(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}\mathbf{u} - \Pi_{j,m}\mathbf{u})\right)\right\|_{L^1(\partial\Omega)} \\ \leq \left\|\Pi_{j+1,k}\mathbf{u} - \Pi_{j,m}\mathbf{u}\right\|_{L^\infty(B_{j,m})} \mathcal{H}^{n-1}(\partial\Omega \cap B_{j+1,k} \cap B_{j,m}).\end{aligned}$$

By the Ahlfors regularity and Poincaré's inequality imply

$$\left\|\mathrm{tr}\left(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}\mathbf{u} - \Pi_{j,m}\mathbf{u})\right)\right\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}\mathbf{u}|\left(\Omega(B^\sharp_{j+1,k}, B^\sharp_{j,m})\right).$$

Summing over k and m and implies

$$\|\mathrm{tr}(T_{j+1}\mathbf{u}) - \mathrm{tr}(T_j\mathbf{u})\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}\mathbf{u}|(U_{j-k_0} \setminus U_{j+k_0}).$$

Ideas of the proof (4)

Finally, we have that

$$\operatorname{tr}(T_{j_0}\mathbf{u}) + \sum_{j\geq j_0}\big(\operatorname{tr}(T_{j+1}\mathbf{u}) - \operatorname{tr}(T_j\mathbf{u})\big) = \lim_{j\to\infty}\operatorname{tr}(T_j\mathbf{u}).$$

is well defined in $L^1(\partial\Omega)$. Hence,

$$\begin{aligned}\Big\|\lim_{j\to\infty}\operatorname{tr}(T_j\mathbf{u})\Big\|_{L^1(\partial\Omega)} &\leq \big\|\operatorname{tr}(T_{j_0}\mathbf{u})\big\|_{L^1} + \sum_{j\geq j_0}\big\|\operatorname{tr}(T_{j+1}\mathbf{u}) - \operatorname{tr}(T_j\mathbf{u})\big\|_{L^1} \\ &\lesssim \|\mathbf{u}\|_{L^1(U_{j_0-k_0}\setminus U_{j_0+k_0})} + \sum_{j\geq j_0}|\mathbb{A}\mathbf{u}|(U_{j-k_0}\setminus U_{j+k_0}) \\ &\lesssim \|\mathbf{u}\|_{L^1(\Omega)} + |\mathbb{A}\mathbf{u}|(\Omega).\end{aligned}$$

Linear growth functionals on $\mathrm{BV}^{\mathbb{A}}(\Omega)$

Breit, Diening, Gmeineder (2017): Functionals on $\mathbf{u} \in \mathrm{BV}^{\mathbb{A}}(\Omega)$

$$\overline{\mathfrak{F}}_{\mathbf{u}_0}[\mathbf{u}] := \int_\Omega f\Big(x, \frac{\mathrm{d}\mathbb{A}\mathbf{u}}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n + \int_\Omega f^\infty\Big(x, \frac{\mathrm{d}D\mathbf{u}}{\mathrm{d}|\mathbb{A}^s\mathbf{u}|}\Big)\,\mathrm{d}|\mathbb{A}^s\mathbf{u}| + \int_{\partial\Omega} f^\infty\Big(x, \nu_{\partial\Omega} \otimes_{\mathbb{A}} \mathrm{tr}(\mathbf{u}-\mathbf{u}_0)\Big)\,\mathrm{d}\mathcal{H}^{n-1}.$$

- Existence of minimizer (representation with trace-term);
- infima coincide: $\inf_{\mathbf{u}\in\mathbf{u}_0+W_0^{\mathbb{A},1}(\Omega)} \mathfrak{F}[\mathbf{u}] = \min_{\mathbf{u}\in\mathrm{BV}^{\mathbb{A}}(\Omega)} \overline{\mathfrak{F}}_{\mathbf{u}_0}[\mathbf{u}]$;
- set of *generalised minimisers* of $\mathfrak{F}$ given by

$$\mathrm{GM}_{\mathbf{u}_0}(\mathfrak{F}) := \left\{\mathbf{u} \in \mathrm{BV}^{\mathbb{A}}(\Omega) : \begin{array}{c}\mathbf{u} \text{ is the } L^1-\text{limit of some} \\ \text{min. seq. } (u_k) \subset \mathbf{u}_0 + W_0^{\mathbb{A},1}(\Omega)\end{array}\right\}$$

coincides with the class of $\mathrm{BV}^{\mathbb{A}}$-minimizers.