Error analysis for 2D stochastic Navier–Stokes equations in bounded domains

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August 2022

Stochastic Navier–Stokes equations

Velocity field $\mathbf{u}$ and pressure π on $Q = (0,T) \times \mathcal{O}$

$$\begin{cases} \mathrm{d}\mathbf{u} = [\mu\Delta\mathbf{u} - \mathrm{div}(\mathbf{u}\otimes\mathbf{u}) - \nabla\pi]\,\mathrm{d}t + \mathbb{G}\,\mathrm{d}W & \text{in } Q, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } Q. \end{cases}$$

- Cylindrical Wiener process W on a separable Hilbert space with basis $(e_k)_{k\geq 0}$ (e.g. $L^2(\mathcal{O})$): $W = \sum_{k\geq 0} \beta_k e_k$;
- Coefficient $\mathbb{G}$ is a Hilbert-Schmidt operator;
- Multiplicative noise $\mathbb{G} = \mathbb{G}(\mathbf{u})$ has to be Lipschitz;
- Defined on filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$;
- Initial datum $\mathbf{u}_0$ can be random.

Stochastic perturbations

1. It can be understood as turbulence in the fluid motion. Holm: Dynamics of fluid particles contain turbulent part $\boldsymbol{\sigma} \circ \mathrm{d}W$

$$\mathrm{d}\mathbf{u} = \ldots + [\boldsymbol{\sigma}\nabla\mathbf{u}] \circ \mathrm{d}W;$$

2. Regularisation by noise. Flandoli-Luo (2021): transport noise delays vorticity blow-up

$$\mathrm{d}\boldsymbol{\omega} = \ldots + [\boldsymbol{\sigma}\nabla\boldsymbol{\omega}] \circ \mathrm{d}W;$$

3. Can be interpreted as a perturbation from the physical model.
4. Apart from the force $\mathbf{f}$, which we are observing, there are further quantities with (usually small) influence on the motion.

Weak formulation

Find $\mathbf{u}$ such that for all $\varphi \in W^{1,2}_{0,\mathrm{div}}(\mathcal{O})$ and all $t \in [0, T]$

$$\int_{\mathcal{O}} \mathbf{u}(t) \cdot \varphi \,\mathrm{d}x = \int_{\mathcal{O}} \mathbf{u}_0 \cdot \varphi \,\mathrm{d}x + \int_0^t \int_{\mathcal{O}} \mathbf{u} \otimes \mathbf{u} : \nabla \varphi \,\mathrm{d}x \,\mathrm{d}\sigma - \mu \int_0^t \int_{\mathcal{O}} \nabla \mathbf{u} : \nabla \varphi \,\mathrm{d}x \,\mathrm{d}\sigma + \int_{\mathcal{O}} \varphi \cdot \int_0^t \mathbb{G} \,\mathrm{d}W \,\mathrm{d}x.$$

- Pressure disappears in weak formulation;
- By Itô-isometry if $\mathbb{G} \in L_2(L^2_x; L^2_x)$

$$\mathbb{E} \left\| \int_0^T \mathbb{G} \,\mathrm{d}W \right\|^2_{L^2_x} = \mathbb{E} \int_0^T \|\mathbb{G}\|^2_{L_2(L^2_x;L^2_x)} \,\mathrm{d}t = \sum_{k \geq 0} \mathbb{E} \int_0^T \|\mathbb{G} e_k\|^2_{L^2_x} \,\mathrm{d}t.$$

Weak pathwise solutions

A stochastically strong solution

is an $(\mathfrak{F}_t)$-adapted stochastic process $\mathbf{u}$ with

$$\mathbf{u} \in C([0,T]; L^2_{\mathrm{div}}(\mathcal{O})) \cap L^2(0,T; W^{1,2}_{0,\mathrm{div}}(\mathcal{O}))$$

$\mathbb{P}$-a.s. which solves the momentum equation in the weak sense.

- Exists on a given stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P})$;
- W is a given $(\mathfrak{F}_t)$-cylindrical Wiener process;
- We have $\mathbf{u}(0,\cdot) = \mathbf{u}_0$ $\mathbb{P}$-a.s. where $\mathbf{u}_0$ is $\mathfrak{F}_0$-measurable with $\mathbf{u}_0 \in L^2(\Omega; L^2_{\mathrm{div}}(\mathcal{O}))$;
- First results by Capiński and Capiński-Cutland (1991/1993).

Qualitative properties: periodic case

Solution $\mathbf{u}$ satisfies for all $r < \infty$

$$\mathbb{E}\Big[\sup_{0\le t\le T}\int_{\mathbb{T}^2}|\nabla \mathbf{u}|^2 + \int_0^T\int_{\mathbb{T}^2}|\nabla^2\mathbf{u}|^2\Big]^r \le c_r\,\mathbb{E}\big[1+\|\mathbf{u}_0\|^2_{W^{1,2}_x}\big]^r.$$

- "Test" by $\Delta\mathbf{u}$ (apply Itô's formula to $t\mapsto\|\mathbf{u}\|^2_{W^{1,2}_x}$);
- Use $\int_{\mathbb{T}^2}(\nabla\mathbf{u})\mathbf{u}\cdot\Delta\mathbf{u}\,\mathrm{d}x = 0$;
- Higher order estimates: $\mathbf{u}_0\in W^{k,2}_x \Rightarrow$ $\mathbf{u}\in L^\infty_t(W^{k,2}_x)\cap L^2_t(W^{k+1,2}_x)$;

Qualitative properties: Dirichlet case

Solution $\mathbf{u}$ satisfies for all $r < \infty$

$$\mathbb{E}\bigg[\sup_{0\leq t\leq \mathfrak{t}_R}\int_{\mathcal{O}}|\nabla\mathbf{u}|^2+\int_0^{\mathfrak{t}_R}\int_{\mathcal{O}}|\nabla^2\mathbf{u}|^2\bigg]^r\leq c(r,R)\,\mathbb{E}\Big[1+\|\mathbf{u}_0\|^2_{W^{1,2}_x}\Big]^r.$$

- Here $\mathfrak{t}_R$ is a stopping time s.t. $\|\nabla u(\mathfrak{t}_R)\|_{L^2_x}\geq R$;
- $\mathfrak{t}_R\to\mathfrak{t}$ with $\mathbb{P}(\mathfrak{t}=\infty)=1$;
- Higher order estimates (depending on R) accordingly.
- Not known for ANY finite deterministic T.

Finite-element spaces

$\mathscr{T}_h$ is quasi-uniform subdivision of $\mathbb{T}^2$ into triangles $\mathcal{S}$, $i, j \in \mathbb{N}$

$$V^h(\mathbb{T}^2) := \{\mathbf{v}_h \in W^{1,2}(\mathbb{T}^2) \,:\, \mathbf{v}_h|_{\mathcal{S}} \in \mathscr{P}_i(\mathcal{S}) \;\forall \mathcal{S} \in \mathscr{T}_h\},$$
$$P^h(\mathbb{T}^2) := \{\pi_h \in L^2(\mathbb{T}^2) \,:\, \pi_h|_{\mathcal{S}} \in \mathscr{P}_j(\mathcal{S}) \;\forall \mathcal{S} \in \mathscr{T}_h\}.$$

- $V^h(\mathbb{T}^2)$ and $P^h(\mathbb{T}^2)$ linked by inf-sup condition:

$$\sup_{\mathbf{v}_h \in V^h(\mathbb{T}^2)} \frac{\int_{\mathcal{O}} \operatorname{div} \mathbf{v}_h \, \pi_h \, \mathrm{d}x}{\|\nabla \mathbf{v}_h\|_{L^2_x}} \geq C \, \|\pi_h\|_{L^2_x} \quad \forall \pi_h \in P^h(\mathbb{T}^2);$$

- Discretely solenoidal finite element functions by

$$V^h_{\mathrm{div}}(\mathbb{T}^2) := \left\{\mathbf{v}_h \in V^h(\mathbb{T}^2) : \int_{\mathbb{T}^2} \operatorname{div} \mathbf{v}_h \; \pi_h \, \mathrm{d}x = 0 \, \forall \pi_h \in P^h(\mathbb{T}^2)\right\}.$$

The algorithm

Find r.v. $\mathbf{u}_{h,m}$ with values in $V^h_{\mathrm{div}}(\mathbb{T}^2)$ s.t. for all $\varphi \in V^h_{\mathrm{div}}(\mathbb{T}^2)$

$$\int_{\mathbb{T}^2} \mathbf{u}_{h,m} \cdot \varphi \, \mathrm{d}x + \tau \int_{\mathbb{T}^2} \big((\nabla \mathbf{u}_{h,m}) \mathbf{u}_{h,m-1} + (\mathrm{div}\, \mathbf{u}_{h,m-1}) \mathbf{u}_{h,m} \big) \cdot \varphi \, \mathrm{d}x$$
$$+ \mu \tau \int_{\mathbb{T}^2} \nabla \mathbf{u}_{h,m} : \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{T}^2} \mathbf{u}_{h,m-1} \cdot \varphi \, \mathrm{d}x$$
$$+ \int_{\mathbb{T}^2} \mathbb{G}(\mathbf{u}_{h,m-1}) \, \Delta_m W \cdot \varphi \, \mathrm{d}x \, .$$

- Initial datum $\mathbf{u}_{h,0} \in V^h_{\mathrm{div}}(\mathbb{T}^2)$ given (e.g. $\mathbf{u}_{h,0} = \Pi_h \mathbf{u}_0$);
- Here $\Delta_m W = W(t_m) - W(t_{m-1})$ where $t_m = m\frac{T}{M}$;
- First analysed by Carelli-Prohl (2012).

Convergence rates (1)

Breit-Dodgson (Num. Math. 2021): for any $\xi > 0$, $\alpha < \frac{1}{2}$, $\beta < 1$

$$\mathbb{P}\left[\frac{\max_m \|\mathbf{u}(t_m) - \mathbf{u}_{h,m}\|^2_{L^2_x} + \sum_m \tau \|\mathbf{u}(t_m) - \mathbf{u}_{h,m}\|^2_{W^{1,2}_x}}{h^{2\beta} + \tau^{2\alpha}} > \xi\right] \to 0$$

- Convergence in probability with rates (almost) 1/2 and 1.
- Carelli-Prohl (2012): same estimate with $h^{2\beta} + \tau^{\alpha}$.
- Low time-regularity of pressure $\rightsquigarrow$ decompose into deterministic and stochastic part.
- Bessaih-Millet: logarithmic L^2-convergence by exponential moment bounds.

Convergence rates (2)

Strategy of the proof:

- Estimates localised in sample space on $\Omega^{\tau,h}$, where $\Omega^{\tau,h}$ with $\mathbb{P}(\Omega^{\tau,h}) \to 1 \rightsquigarrow$ estimate for $\mathbb{E}[\mathbb{I}_{\Omega^{\tau,h}} \dots]$
- $\Omega^{\tau,h}$ controls blow-up of $\max_m \|\nabla \mathbf{u}(t_m)\|_{L^2_x}$
- Introduce in m-th step $\Omega^{\tau,h}_{m-1}$, multiply by $\mathbb{I}_{\Omega^{\tau,h}_{m-1}}$ to control blow-up of $\max_{n \le m-1} \|\nabla \mathbf{u}(t_n)\|_{L^2_x}$;
- Use regularity estimates for continuous solution.

Multiplicative noise

Breit-Prohl (Preprint): for any $\xi > 0$, $\alpha < \frac{1}{2}$, $\beta < 1$

$$\mathbb{P}\left[\frac{\max_m \|\mathbf{u}(\mathfrak{t}_m^R) - \mathbf{u}_{h,m}\|_{L_x^2}^2 + \sum_m \tau \|\mathbf{u}(\mathfrak{t}_m^R) - \mathbf{u}_{h,m}\|_{W_x^{1,2}}^2}{h^{2\beta} + \tau^{2\alpha}} > \xi\right] \to 0$$

- Multiplication by $\mathbb{I}_{\Omega_{m-1}^{\tau,h}}$ insufficient to control continuous solution until t_m!
- Discrete stopping times $\mathfrak{t}_m^R \leq \mathfrak{t} \rightsquigarrow (\mathfrak{F}_{t_m})$-stopping times but not $(\mathfrak{F}_t)$-stopping times;
- Since $\mathfrak{t}_m^R \to t_m$ in probability it does not effect convergence result.

Additive noise (1)

Consider $\mathbf{y}(t) = \mathbf{u}(t) - \int_0^t \Phi \,\mathrm{d}W(s) = \mathbf{u}(t) - \Phi W(t)$ which solves

$$\partial_t \mathbf{y} = \mu \mathcal{A}\mathbf{y} - \mathcal{P}\big[(\nabla \mathbf{y})\mathbf{y}\big] + \mu \mathcal{A}[\Phi W] - \mathcal{P}\big[\mathcal{L}^W(\mathbf{y})\big]$$

- $\mathbf{y}$ solves deterministic equation with random coefficients;
- $\mathbf{y}$ has a weak time-derivative;
- Control temporal error between $\mathbf{y}(t_m)$ and $\mathbf{y}_m$ using stopping times for $\mathbf{u}$, $(\mathbf{u}_m)_{m=1}^M$ and ΦW;
- Error between $\mathbf{u}$ and $\mathbf{y}$ known.

Additive noise (2)

Breit-Prohl (IMAJNA, online): for any $\xi > 0$, $\alpha, \beta < 1$

$$\max_m \mathbb{P}\left[\frac{\|\mathbf{y}(t_m^R) - \mathbf{y}_{h,m}\|^2_{L^2_x} + \sum_m \tau \|\mathbf{y}(t_m^R) - \mathbf{y}_{h,m}\|^2_{W^{1,2}_x}}{h^{2\beta} + \tau^{2\alpha}} > \xi\right] \to 0$$

- Convergence of (up to) order 1 in time!
- Control temporal error between $\mathbf{y}_m$ and $\mathbf{y}_{h,m}$ using estimates for $(\mathbf{y}_m)_{m=1}^M$.
- Bessaih-Millet: Convergence rates in mean-square for $\mu \gg 1$.