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J. Brasche

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Institut für Mathematik, Erzstraße 1, 38678 Clausthal-Zellerfeld, Tel. (05323) 72-2413

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1 Introduction

Let U be an open subset of $\mathbb{R}^n$ with Lipschitz boundary F and Γ the closure of U. By Kato's monotone convergence theorem (cf. [10]), the operators $(-\Delta + b1_\Gamma + 1)^{-1}$ in $L^2(\mathbb{R}^n)$ converge strongly to some operator L_Γ, as b tends to infinity. Bruneau, Carbou, Demuth, Kirsch, Mc Gillivray et al. (cf. [5, 7, 8] and references given therein) have presented conditions which are sufficient in order that these operators even converge w.r.t. the operator norm. In addition, they have derived estimates for the rate of convergence.

Brasche and Demuth have studied perturbations of the free Hamiltonian by potentials supported by a set Γ with Lebesgue measure zero [4]. Of course, in this case the operator of multiplication by the function 1_Γ is identically equal to zero. Therefore they have used measures μ_Γ supported by Γ instead of the characteristic function 1_Γ as the perturbing potential. It turned out that the investigation of such singular potentials required new techniques and that these new techniques were suitable in order to treat a fairly general class of large coupling approximation problems. Ben Amor and Brasche have continued these investigations [2]. Let us briefly describe the result of [2] relevant for this short note.

Let $\mathcal{E}$ be a densely defined nonnegative closed quadratic form in the Hilbert space $\mathcal{H}$ and H the nonnegative self-adjoint operator associated to $\mathcal{E}$. Let $\mathcal{P}$ be a nonnegative quadratic form such that

$$D(\mathcal{E} + \mathcal{P}) := D(\mathcal{E}) \cap D(\mathcal{P}) \supset D(H) \tag{1}$$

and $\mathcal{E} + b\mathcal{P}$ is closed for one and therefore every $b > 0$. For every $b > 0$ let H_b be the nonnegative self-adjoint operator in $\mathcal{H}$ associated to $\mathcal{E} + b\mathcal{P}$.

By Kato's monotone convergence theorem, the operators $(H_b+1)^{-1}$ converge strongly to an operator L. Under additional assumptions one even obtains convergence w.r.t. the operator norm: Choose an auxiliary Hilbert space

$\mathcal{H}_{aux}$ and a closed operator J from $(D(\mathcal{E}),\mathcal{E}_1)$ to $\mathcal{H}_{aux}$ such that

$$D(J)=D(\mathcal{E})\cap D(\mathcal{P}) \text{ and } \mathcal{P}(u,u)=\| Ju \|^2_{aux} \quad \forall\, u\in D(J)$$

and ran$J$ is dense in $\mathcal{H}_{aux}$. Here $\mathcal{E}_1(u,u):=\mathcal{E}(u,u)+\| u \|^2$ for all $u\in D(\mathcal{E})$. JJ^* is an invertible nonnegative self-adjoint operator in $\mathcal{H}_{aux}$. Let

$$\check{H}:=(JJ^*)^{-1}.$$

Let $r>0$. There exists a finite constant c_r such that

$$\| (H_b+1)^{-1}-L \| \leq \frac{c_r}{b^r} \quad \forall\, b>0 \tag{2}$$

provided

$$J(D(H))\subset D(\check{H}^s), \text{ where } s=\frac{1}{2}+\frac{r}{2} \tag{3}$$

([2], Proposition 2).

$D(H_b)=W^{2,2}(\mathbb{R}^n)$ and $H_b u=-\Delta u+b1_\Gamma u$ for every $u\in D(H_b)$, if $\mathcal{H}=L^2(\mathbb{R}^n)$, $D(H)=W^{2,2}(\mathbb{R}^n)$, $Hu=-\Delta u$ for every $u\in D(H)$ and

$$D(\mathcal{P})=L^2(\mathbb{R}^n), \quad \mathcal{P}(u,u)=\int_\Gamma |u|^2dx \quad \forall\, u\in L^2(\mathbb{R}^n).$$

In this case

$$D(\mathcal{E})=W^{1,2}(\mathbb{R}^n), \quad \mathcal{E}(u,u)=\int |\nabla u|^2dx \quad \forall\, u\in W^{1,2}(\mathbb{R}^n) \tag{4}$$

and we can choose $\mathcal{H}_{aux}$ and J as follows:

$$\mathcal{H}_{aux}=L^2(\Gamma), \quad Ju=u\restriction\Gamma \quad \forall\, u\in W^{1,2}(\mathbb{R}^n). \tag{5}$$

With this choice of $\mathcal{H}_{aux}$ and J we get, as a special case of a general result by Ben Amor ([1], formula (4.5)), that $\check{H}$ is the nonnegative self-adjoint operator in $L^2(\Gamma)$ associated to the following quadratic form:

$$\begin{aligned} D(\check{\mathcal{E}}_1) &:= J(W^{1,2}(\mathbb{R}^n)), \\ \check{\mathcal{E}}_1(Ju,Ju) &:= \langle Pu,Pu\rangle_{W^{1,2}} \quad \forall\, u\in W^{1,2}(\mathbb{R}^n), \end{aligned} \tag{6}$$

where Pu denotes the unique element of $W^{1,2}(\mathbb{R}^n)$ satisfying

$$JPu = Ju \text{ and } (-\Delta + 1)Pu = 0 \text{ on } \mathbb{R}^n \setminus \Gamma. \tag{7}$$

Note that $Pu = Pv$, if $Ju = Jv$, so that the above definition (6) is correct.

$\check{\mathcal{E}}_1$ is called the trace of the Dirichlet form $\mathcal{E}_1$ w.r.t. the measure $1_\Gamma\, dx$ (cf. [9], (6.2.4)). Chen, Fukushima and Ying have shown that there exist positive Radon measures k and J such that

$$\begin{aligned} \check{\mathcal{E}}_1(u,u) \quad = \quad & \int_\Gamma (|u|^2 + |\nabla u|^2)dx + \int_F |\tilde{u}|^2 dk \\ & + \int_{F\times F} |\tilde{u}(x) - \tilde{u}(y)|^2 J(dx\,dy) \quad \forall\, u \in D(\check{\mathcal{E}}_1) \end{aligned} \tag{8}$$

($\tilde{u}$ denotes any quasi continuous representative of u) and derived a representation of the killing measure k and the jumping measure J in terms of the stochastic process generated by $-\Delta + 1$ (cf. [6], actually the results by Chen et al. are much more general).

Let $s > 1/2$. Often it is fairly simple to check whether

$$J(W^{2,2}(\mathbb{R}^n)) \subset D((-\Delta_N)^s) \tag{9}$$

($-\Delta_N$ denotes the Neumann-Laplacian on U). If (9) holds and, in addition,

$$D((-\Delta_N)^s) \subset D(\check{H})^s) \tag{10}$$

then, by (2) and (3), there exists a finite constant c_r such that

$$\| \, (-\Delta + b1_\Gamma + 1)^{-1} - L_\Gamma \, \| \leq \frac{c_r}{b^r} \quad \forall\, b > 0, \text{ where } s = \frac{1}{2} + \frac{r}{2}. \tag{11}$$

In this short note we shall show for a large class of self-adjoint realizations H_A of the Laplacian on U that there exist $s > 1/2$ such that

$$D((-\Delta_N)^s) \subset D((H_A)^s). \tag{12}$$

2 Hypothesis and notation

- U is an open non-empty subset of $\mathbb{R}^n$ with Lipschitz boundary F and $\Gamma = \bar{U}$
- $-\Delta_N$ is the Neumann-Laplacian in $L^2(U)$
- μ is a positive Radon measure on F and A a nonnegative bounded self-adjoint operator in $L^2(F, \mu)$
- If $B = B^* \geq c > 0$, then we define for every $\tau \in \mathbb{R}$

$$\| u \|_{B^\tau} := \| B^\tau u \|,$$

denote by $\tilde{D}(B^\tau)$ the completion of $D(B^\tau)$ w.r.t. this norm (we choose the same notation for the norm on the completion) and by B_τ^{-1} the canonical isometry from $\tilde{D}(B^{\tau-1})$ onto $\tilde{D}(B^\tau)$

- $\alpha > 0$ is a real number and $J_F : D((-\Delta_N + \alpha)^{1/2}) \longrightarrow L^2(F, \mu)$ a bounded linear mapping satisfying

$$\| J_F \| \| A \| < 1 \tag{13}$$

- There exists $t \in (0, 1/2)$ such that J_F can be extended to a bounded mapping J_e from $D((-\Delta_N + \alpha)^t)$ to $L^2(F, \mu)$
- In what follows we fix α and t with the mentioned properties and put

$$s := 1 - t$$

- H_A denotes the nonnegative self-adjoint operator in $L^2(\Gamma)$ satisfying

$$\begin{aligned} D((H_A + \alpha)^{1/2}) &= D((-\Delta_N + \alpha)^{1/2}), \\ \| (H_A + \alpha)^{1/2} u \|^2 &= \| (-\Delta_N + \alpha)^{1/2} u \|^2 + \| AJ_F u \|^2_{L^2(F,\mu)} \end{aligned} \tag{14}$$

- $G_A := (H_A + \alpha)^{-1}$

3 Local and non-local boundary conditions

Lemma 1 *With the notation and under the hypothesis of the previous section the following holds true.*

$$D((-\Delta_N+\alpha)^s)\subset D((H_A+\alpha)^s). \tag{15}$$

Remark: Actually we have equality in (15).

Proof: By definition,

$$D((-\Delta_N+\alpha)^{1/2})=D((H_A+\alpha)^{1/2})$$

and, by (13), the norms $\|\cdot\|_{(-\Delta_N+\alpha)^{1/2}}$ and $\|\cdot\|_{(H+\alpha)^{1/2}}$ are equivalent. By interpolation, we get the corresponding statements for every $\tau\in[0,1/2]$ and passing to the dual spaces for every $\tau\in[-1/2,0]$. Hence

$$\begin{aligned}\tilde D((-\Delta_N+\alpha)^\tau) &= \tilde D((H_A+\alpha)^\tau) \quad \forall\,\tau\in[-1/2,1/2],\\ \|\cdot\|_{(-\Delta_N+\alpha)^\tau} &\sim \|\cdot\|_{(H_A+\alpha)^\tau} \quad \forall\,\tau\in[-1/2,1/2].\end{aligned} \tag{16}$$

Let $u\in D((-\Delta_N+\alpha)^s)$. There exists $w\in\tilde D((-\Delta_N+\alpha)^{s-1})$ such that

$$u=(-\Delta_N+\alpha)_s^{-1}w.$$

By (16) and since

$$u=\left((-\Delta_N+\alpha)_s^{-1}-(H_A+\alpha)_s^{-1}\right)w+(H_A+\alpha)_s^{-1}w,$$

we only need to show that $(-\Delta_N+\alpha)_s^{-1}-(H_A+\alpha)_s^{-1}$ maps the space $\tilde D((H+\alpha)^{s-1})$ into $D((H+\alpha)^s)$.

We have

$$(-\Delta_N+\alpha)^{-1}-(H_A+\alpha)^{-1}=(J_FG_A)^*A\,(1-AJ_FJ_F^*A)^{-1}\,AJ_FG_A \tag{17}$$

(cf. [3], Theorem 3).

- G_A can be uniquely extended to a mapping G_s from $\tilde{D}((H_A+\alpha)^{s-1})$ into $D((H_A+\alpha)^s)$
- Since $s \geq t$, J_F is, in particular, a bounded mapping from $D((H+\alpha)^s)$ to $L^2(F,\mu)$
- $A\,(1 - AJ_F J_F^* A)^{-1}\,A$ is a bounded operator in $L^2(F,\mu)$
- By hypothesis, J_F is a bounded operator from $D((-\Delta_N+\alpha)^t)$ into $L^2(F,\mu)$ and therefore, by (16), also a bounded operator from $D((H_A+\alpha)^t)$ into $L^2(F,\mu)$. This implies that $J_F G_A^t$ is a bounded mapping from $L^2(\Gamma)$ into $L^2(F,\mu)$. Since $(J_F G_A)^* = G_A^s (J_F G_A^t)^*$ this implies that $(J_F G_A)^*$ is a bounded mapping from $L^2(F,\mu)$ into $D((H_A+\alpha)^s)$.

It follows that $(-\Delta_N+\alpha)^{-1} - (H_A+\alpha)^{-1}$ can be extended to a bounded mapping D_e from $\tilde{D}((H_A+\alpha)^{s-1})$ into $D((H_A+\alpha)^s)$.

D_e is, in particular, a bounded mapping from $\tilde{D}((H_A+\alpha)^{s-1})$ into $D((H_A+\alpha)^{1/2})$ and equals $(-\Delta_N+\alpha)^{-1} - (H_A+\alpha)^{-1}$ on $L^2(\Gamma)$. On the other hand, by (16), $(-\Delta_N+\alpha)_s^{-1} - (H_A+\alpha)_s^{-1}$ is also a bounded mapping from $\tilde{D}((H_A+\alpha)^{s-1})$ into $D((H_A+\alpha)^{1/2})$ and equals $(-\Delta_N+\alpha)^{-1} - (H_A+\alpha)^{-1}$ on $L^2(\Gamma)$. Thus $D_e = (-\Delta_N+\alpha)_s^{-1} - (H_A+\alpha)_s^{-1}$. □

Corollary 2 *Let k and J be the measures occurring in the representation (8) of the quadratic form $\check{\mathcal{E}}_1$ and $\check{H}$ the nonnegative self-adjoint operator associated with $\check{\mathcal{E}}_1$. Suppose that there exist $s \in (1/2,1)$ and $\alpha, c, c_1, c_2 < \infty$ such that the following holds:*

- $$\int |\tilde{u}|^2 dk \leq c\langle u, (-\Delta_N+\alpha)^{2-2s} u\rangle \quad \forall\, u \in D(-\Delta_N)$$
- $$\int_F |\tilde{u}|^2 dk + \int_{F\times F} |\tilde{u}(x) - \tilde{u}(y)|^2 J(dx\,dy) \leq c_2 \int_F |\tilde{u}|^2 dk \quad \forall\, u \in D(\check{\mathcal{E}}_1),$$
  $$c_1 c_2 < 1 \text{ and } \int |\tilde{u}|^2 dk \leq c_1 \langle u, (-\Delta_N+\alpha)u\rangle \quad \forall\, u \in D(-\Delta_N).$$

Then

$$D((-\Delta_N + \alpha)^s) \subset D((\check{H} + \alpha)^s)).$$

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