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# Further examples to a question of Atiyah

M. Neuhauser

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Institut für Mathematik, Erzstraße 1, 38678 Clausthal-Zellerfeld, Tel. (05323) 72-2413

# Further examples to a question of Atiyah

Markus Neuhauser

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**Abstract**

This note presents how the computation of the spectrum of a multiple of the Markov operator of the simple random walk on the Cayley graph of an HNN-extension of the lamp-lighter group from [2] can be used to obtain new examples that the answer to a strong form of a question formulated by Atiyah [1] is negative. Apparently there seems to be only one other known example. A positive answer would have predicted the values of von Neumann dimensions of the kernels of operators corresponding to multiplication with matrices whose entries are group algebra elements on powers of Hilbert spaces of square-summable functions on the group. The original question of Atiyah is apparently not answered up to now.

For a finitely generated group $\Gamma$ right multiplication with $A \in \mathbf{C}\Gamma$ on $\mathbf{C}\Gamma$ induces a bounded linear operator on the Hilbert space $\ell^2(\Gamma)$. Denote this operator also by $A$. In the sequel $A$ is self-adjoint. In this case $A = \int \sigma \, \mathrm{d}E(\sigma)$ where $E$ is a spectral measure on $\mathbf{R}$. Then $E(\{0\}) : \ell^2(\Gamma) \to \ell^2(\Gamma)$ is the orthogonal projection onto $\ker(A)$. Let $e$ denote the neutral group element of $\Gamma$. Consider $e$ also as the corresponding element of $\ell^2(\Gamma)$. Let $\mu(B) = \langle E(B)e, e\rangle$ then $\int \sigma \, \mathrm{d}\mu(\sigma) = \langle eA, e\rangle$. Define now the *von Neumann dimension*:

$$\dim_\Gamma(\ker A) = \mu(\{0\})$$

Let $\mathrm{fin}(\Gamma) = \{H \leq \Gamma : |H| < \infty\}$ be the set of all finite subgroups of $\Gamma$ and $F_\Gamma = \sum_{H \in \mathrm{fin}(\Gamma)} |H|^{-1} \mathbf{Z}$ an additive subgroup of $\mathbf{Q}$. If $\Gamma$ is a group that satisfies the strong form of the question of Atiyah this would imply that for $A \in \mathbf{C}\Gamma$ holds $\dim_\Gamma(\ker A) \in F_\Gamma$ see for example [5, page 369]. The following presents new examples for which this does not hold.

**Theorem 1** *Let $q \geq 2$ and the group $\Gamma$ be given by the set of affine matrices*

$$\left\{ \begin{pmatrix} x^\alpha (x+1)^\beta & f \\ 0 & 1 \end{pmatrix} : \alpha, \beta \in \mathbf{Z}, f \in (\mathbf{Z}/q\mathbf{Z}) \left[x^{-1}, (x+1)^{-1}, x\right] \right\}$$

*then $\Gamma$ is metabelian and in particular elementary amenable. The order of every finite subgroup of $\Gamma$ is a divisor of a power of $q$. There exists an $A \in \mathbf{C}\Gamma$ such that:*

$$\dim_{\Gamma}(\ker A) = \frac{2(q-1)}{(q^2+1)(q+1)^2}$$

The denominator $(q^2+1)(q+1)^2$ is coprime to $q$. For example for $q=2$ the von Neumann dimension in Theorem 1 results in $2/45$. Hence these are examples that do not satisfy the preceding conclusion. The first such example was presented in [4]. But in contrast to that the spectrum of the operator $A$ here is determined on $\ell^2(\Gamma)$ and not only on the canonically embedded subspace ismorphic to $\ell^2(H)$ where $H$ is a subgroup of $\Gamma$.

In [3] values of $\dim_{\Gamma}(\ker A)$ are computed for which it is conceivable but not proven that they are not rational. In case this is true it would provide a negative answer to Atiyah's question from [1].

In terms of analytic $L^2$-Betti numbers $b_p^{(2)}(M,g)$ introduced by Atiyah which measure the size of the space of harmonic square-integrable $p$-forms on the universal covering $\widetilde{M}$ of a closed Riemannian manifold $(M,g)$ the strong form of the question of Atiyah can be formulated in the following way see for example [4]: If $M$ has $\Gamma$ as its fundamental group then $b_p^{(2)}(M,g) \in F_{\Gamma}$.

As in [4] the following result can be obtained which is proven from Theorem 1 in exactly the same way as there.

**Theorem 2** *Let $q \geq 2$. There exists a smooth oriented closed Riemannian manifold $(M,g)$ of dimension 7 with $\pi_1(M) = \Gamma$ such that for the third $L^2$-Betti number holds*

$$b_3^{(2)}(M,g) = \frac{2(q-1)}{(q^2+1)(q+1)^2}$$

*where $b_3^{(2)}(M,g)$ denotes the combinatorial $L^2$-Betti number of a triangulation of $M$.*

The group $\Gamma$ is now described as a presentation in terms of generators and relations. It is an HNN-extension of the lamp-lighter group see also [2].

**Proposition 3** *The group $\Gamma$ is isomorphic to the group given by the presentation*

$$\left\langle a,t,s : a^q, [t,s], \left[tat^{-1},a\right], atat^{-1}sa^{-1}s^{-1}\right\rangle$$

*where an isomorphism is defined by:*

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} x+1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

**Proof.** These matrices fulfill the relations of $a$, $s$, and $t$ of the presentation of this group and they generate $\Gamma$. Thus this mapping can be extended to an epimorphism from the three generators onto $\Gamma$.

Let $w$ be an element of the group given by the preceding presentation. The relations show that $w$ can be written as $w = rs^{\beta}$ with $\beta \in \mathbf{Z}$ and $r$ an element of the subgroup generated by $a$ and $t$. Suppose that $w$ is mapped to the identity matrix by the defined homomorphism. This entails that $\beta = 0$.

The relations employ that not only $tat^{-1}$ commutes with $a$ but also $t^j at^{-j}$ for all $j \in \mathbf{Z}$ as by induction

$$\begin{aligned} e &= s\left[t^j at^{-j}, a\right] s^{-1} = \left[t^j sas^{-1}t^{-j}, sas^{-1}\right] = \left[t^j atat^{-j-1}, atat^{-1}\right] \\ &= \left[t^j at^{-j}t^{j+1}at^{-j-1}, atat^{-1}\right] = \left[t^{j+1}at^{-j-1}, a\right] \end{aligned}$$

and $e = t^{-j}\left[t^j at^{-j}, a\right] t^j = \left[a, t^{-j}at^j\right]$.

Thus $r$ can be written as $r = t^{\alpha}a^{k_0}ta^{k_1}t\cdots ta^{k_d}t^{\omega}$. This is mapped to

$$\begin{pmatrix} x^{\alpha+d+\omega} & \sum_{j=0}^{d} k_j x^{j+\alpha} \\ 0 & 1 \end{pmatrix}$$

and implies that $\alpha + d + \omega = 0$ and $k_0 \equiv k_1 \equiv \cdots \equiv k_d \equiv 0 \bmod q$. Hence $w = r = e$. This proves that the mapping is one-to-one and hence an isomorphism is obtained. $\square$

It is easy to check that $\Gamma$ is metabelian and that the order of every finite subgroup divides a power of $q$ as claimed in Theorem 1. Indeed let $N$ be the normal subgroup generated by $a$. It holds that $N = \langle\{t^j at^{-j} : j \in \mathbf{Z}\}\rangle \cong \bigoplus_{\mathbf{Z}}(\mathbf{Z}/q\mathbf{Z})$ and $\Gamma/N \cong \langle t, s : [t, s]\rangle \cong \mathbf{Z}^2$. This proves that $\Gamma$ is metabelian. If $H$ is a finite subgroup of $\Gamma$ then $H$ must be a subset of $N$. Thus $H$ is contained in $(\mathbf{Z}/q\mathbf{Z})^h \leq N$ for some $h \in \mathbf{N}$ which entails that the order of $H$ divides $q^h$.

**Theorem 4** *Let*

$$A = \sum_{k=0}^{q-1} a^k (s+t) + \left(s^{-1} + t^{-1}\right) a^k + t^{-1}a^k s + s^{-1}a^k t \in \mathbf{Z}\Gamma$$

*be a multiple of the Markov operator of the simple random walk on $\Gamma$ given by the system of generators consisting of $a^k s, a^k t, t^{-1}a^k s$, $k \in \mathbf{Z}/q\mathbf{Z}$, and their inverses. Then $A$, considered as an operator on $\ell^2(\Gamma)$, has eigenvalues*

$$\lambda_{m_1,m_2,n}, m_1, m_2 > 0\,,\ m_1 + m_2 < n$$

*given by*

$$\begin{aligned}
&\lambda_{m_1,m_2,n} \\
= \;& 2q\left(4\cos\left(\frac{\pi(m_1-m_2)}{3n}\right)\cos\left(\frac{\pi(m_1+2m_2)}{3n}\right)\cos\left(\frac{\pi(2m_1+m_2)}{3n}\right)-1\right)
\end{aligned}$$

*and the* $\Gamma$*-dimension of the eigenspaces corresponding to the eigenvalue* $\sigma$ *is*

$$\dim_\Gamma \ker\left(A-\sigma e\right) = \left(q-1\right)^3 \sum_{n=2}^{\infty} \Lambda_n\left(\sigma\right) q^{-n}$$

*where*

$$\Lambda_n\left(\sigma\right) = \left|\left\{(m_1,m_2) : m_1, m_2 > 0\,,\; m_1+m_2 < n, \lambda_{m_1,m_2,n} = \sigma\right\}\right|$$

*is the number of pairs* $(m_1,m_2)$ *for which* $\lambda_{m_1,m_2,n} = \sigma$*.*

**Proof.** For a proof of the explicit eigenvalues of $(6q)^{-1}A$ see [2, Corollary 5.22] and note [2, Corollary 3.15] and that:

$$\begin{aligned}
& \cos\left(\frac{2\pi(m_1-m_2)}{3n}\right)+\cos\left(\frac{2\pi(m_1+2m_2)}{3n}\right)+\cos\left(\frac{2\pi(2m_1+m_2)}{3n}\right) \\
= \;& 2\left(\cos\left(\frac{\pi(m_1-m_2)}{3n}\right)\right)^2 - 1 + 2\cos\left(\frac{\pi(m_1+m_2)}{n}\right)\cos\left(\frac{\pi(m_1-m_2)}{3n}\right) \\
= \;& 4\cos\left(\frac{\pi(m_1-m_2)}{3n}\right)\cos\left(\frac{\pi(m_1+2m_2)}{3n}\right)\cos\left(\frac{\pi(2m_1+m_2)}{3n}\right)-1
\end{aligned}$$

Observe that the spectral measure is pure point and its value at $\sigma$ is in [2, Section 5.D] computed to be $\left(q-1\right)^3 \sum_{n=2}^{\infty} \Lambda_n\left(\sigma\right) q^{-n}$ as every $(m_1,m_2,n)$ with $\lambda_{m_1,m_2,n} = \sigma$ contributes $\left(q-1\right)^3 q^{-n}$. Since this value is exactly the $\ell^2$-dimension of the corresponding eigenspace, the proof of the theorem is finished. □

In general, it seems not trivial to determine all $(u_1,u_2,v)$ for which $\lambda_{u_1,u_2,v} = \lambda_{m_1,m_2,n}$, where $(m_1,m_2,n)$ is given. Obviously $\lambda_{m_1,m_2,n} = \lambda_{cm_1,cm_2,cn}$ for each $c \in \mathbf{N}$ and $\lambda_{m_1,m_2,n} = \lambda_{m_2,m_1,n}$. At least for $\sigma = -2q$ the value $\dim_\Gamma \ker\left(A-\sigma e\right)$ can be computed explicitly.

**Corollary 5** *There is an* $A \in \mathbf{Z}\Gamma$ *such that for* $A$ *considered as an operator* $A : \ell^2\left(\Gamma\right) \to \ell^2\left(\Gamma\right)$ *holds:*

$$\dim_\Gamma \ker\left(A\right) = \frac{2\left(q-1\right)}{\left(q^2+1\right)\left(q+1\right)^2}$$

**Proof.** Initially take $A$ of Theorem 4 and $\sigma = -2q$.

It holds that $\lambda_{m_1,m_2,n} = -2q$ if and only if $m_1 - m_2$, $m_1 + 2m_2$, or $2m_1 + m_2 \in 3n(1/2 + \mathbf{Z})$. The conditions $m_1, m_2 > 0$ and $m_1 + m_2 < n$ imply $|m_1 - m_2| < n$ and $m_1 + 2m_2, 2m_1 + m_2 < 2n$. Hence $\lambda_{m_1,m_2,n} = -2q$ is equivalent to $m_1 + 2m_2 = 3n/2$ or $2m_1 + m_2 = 3n/2$. Thus $n$ must be even and in the first case $m_1 = 3n/2 - 2m_2$ and in the second $m_2 = 3n/2 - 2m_1$. Consider now the first case. From the constraints $m_1 + m_2 < n$ and $m_1 > 0$ follows $n/2 < m_2 < 3n/4$. This entails $0 < m_1 < n/2$. Interchanging the indices 1 and 2 yields for the second case $n/2 < m_1 < 3n/4$ and $0 < m_2 < n/2$.

The number of pairs $(m_1, m_2)$ fulfilling these conditions is $\Lambda_n(-2q) = n/2 - 2$ if $n$ is a multiple of 4 and $\Lambda_n(-2q) = n/2 - 1$ if $n$ is even but not a multiple of 4. From

$$\begin{aligned}
\sum_{n=1}^{\infty} \Lambda_{2n}(-2q)\, q^{-2n} &= \sum_{n=1}^{\infty} 2nq^{-4n-2} + 2nq^{-4n-4} \\
&= 2q^{-8}\left(q^2+1\right) \sum_{n=1}^{\infty} nq^{-4n+4} \\
&= \frac{2(q^2+1)}{(q^4-1)^2} = \frac{2}{(q^2+1)(q^2-1)^2}
\end{aligned}$$

follows that:

$$\dim_\Gamma \ker(A + 2qe) = \frac{2(q-1)}{(q^2+1)(q+1)^2}$$

Replace now $A$ by $A + 2qe$. This implies that 0 is in the spectrum of $A$ and that $\dim_\Gamma(\ker A) = \frac{2(q-1)}{(q^2+1)(q+1)^2}$. $\square$

This finishes the proof of Theorem 1.

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Institut für Mathematik
TU Clausthal
Erzstr. 1
38678 Clausthal-Zellerfeld
Germany
`neuhauser@math.tu-clausthal.de`