Asymptotic Properties of a Generalized Cross Entropy Optimization Algorithm

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***Abstract*—The discrete cross entropy optimization algorithm iteratively samples solutions according to a probability density on the solution space. The density is adapted to the good solutions observed in the present sample before producing the next sample. The adaptation is controlled by a so-called smoothing parameter. We generalize this model by introducing a flexible concept of feasibility and desirability into the sampling process. In this way, our model covers several other optimization procedures, in particular the ant based algorithms.**

The focus of this paper is on some theoretical properties of these algorithms. We examine the first hitting time τ of an optimal solution and give conditions on the smoothing parameter for τ to be finite with probability one. For a simple test case, we show that runtime can be polynomially bounded in the problem size with a probability converging to 1. We then investigate the convergence of the underlying density and of the sampling process. We show in particular, that a constant smoothing parameter, as it is often used, makes the sample process converge in finite time, freezing the optimization at a single solution which need not be optimal. Moreover, we define a smoothing sequence that makes the density converge without freezing the sample process and that still guarantees the reachability of optimal solutions in finite time. This settles an open question from literature.

***Index Terms*—Evolutionary computation, cross entropy optimization, ant colony optimization, heuristic optimization, discrete optimization, model based optimization.**

I. Introduction

CROSS entropy (CE) optimization is a widely used tool for heuristic optimization in particular for discrete problems, see [1] for an overview. As was noted before (see e.g. [2]), it also has much in common with ant algorithms (ACOs).

In this paper we use a generalized version of the CE algorithm that includes most of the features of ACOs. We concentrate on theoretical properties of the underlying processes, that hold under very general conditions and therefore apply to all types of algorithms covered by our model. The paper is inspired by [3] where for a particular CE model convergence properties were proven.

In a CE model, we are considering a cost function f on a finite set S of solutions. A solution is a tuple of fixed length. We use a probability distribution or density on S as a model for producing good solutions. This density is iteratively adapted with the goal to give an increasing weight to optimal solutions. Roughly, the method proceeds in two steps:

Sampling: We take a sample of fixed size N from S, based on the present distribution $\mathbf{\Pi}_t$, starting with a given density $\mathbf{\Pi}_0$ for $t = 0$ (e.g. the uniform distribution).

Adaptation: We evaluate the solutions in the sample with the cost function f and determine the relative frequencies of components in the best solutions. We define $\mathbf{\Pi}_{t+1}$ by adapting $\mathbf{\Pi}_t$ to these frequencies. Here, a so-called smoothing parameter ϱ_t controls the relative weight of the sample during the adaptation process.

In this way, $\mathbf{\Pi}_t$ is expected to give increasing weight to the better part of the solution space.

In [3], the impact of the smoothing parameter on the convergence of the process $(\mathbf{\Pi}_t)_{t\geq 0}$ is investigated. The authors give conditions on ϱ_t under which an optimal solution is sampled with probability one after finitely many iterations (reachability). In particular, they found that for a constant smoothing parameter $\varrho_t \equiv \varrho$, the distribution $\mathbf{\Pi}_t$ will converge to a one point mass (convergence of density), but that one has to make ϱ arbitrarily small to increase the probability of reachability. Solutions are restricted to 0-1-tuples in [3].

We extend these results in several directions. We allow solutions to be strings of arbitrary elements and do not require the optimal solution to be unique. Most importantly, we introduce a notion of feasibility and desirability to restrict the set of solutions and to bias the sampling. The samples in the Sampling step above are actually drawn according to a mixture of the present distribution $\mathbf{\Pi}_t$ and a given feasibility measure. This also allows to include a greedy aspect into the construction of the solutions, as it is common in ACOs under the name of 'visibility'. However, this additional feature makes the mathematical model much more complex compared to that in [3].

Still, we are able to show that the results from [3] hold for this generalized model. We investigate τ, the number of iterations until an optimal solution is sampled for the first time. We show that, if the smoothing parameter ϱ_t is a constant, then τ has a strictly positive probability to be infinite, and therefore the expected runtime is ∞. To get more insight into the finite time behavior of the algorithm, we consider a standard test problem (LeadingOne, see [4]). We show that for this problem, even if ϱ_t is constant, the runtime, i.e. the number

Manuscript received 28 Nov 2013; revised 16 Apr 2014 and 21 Jun 2014.
The work of Z. Wu was supported by China Scholarship Council, No.2010624003.
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of solutions sampled until an optimal one is found, is bounded by a polynomial in the problem size with a probability that is converging fast against one.

In genetic algorithms, the phenomenon of *genetic drift* is well-known, see e.g. [5], it describes the loss of all variation in the solutions. A similar thing may also happen in our model. We show that, for smoothing parameters ϱ_t bounded away from 0, the distribution $\mathbf{\Pi}_t$ as well as the sampled solutions converge. Thus, for fixed $\varrho_t \equiv \varrho > 0$, the sample process is absorbed into a fixed point after finitely many iterations, and this fix point need not be an optimal solution. This may be the theoretical proof for the stagnation in sampling that has been reported in [6], where CE was applied to a Maximal Cut problem.

In CE, it is of great importance whether almost sure reachability of optimal solutions and convergence of the underlying density are compatible. This important question remained open in [3]. We solve this question by giving smoothing sequences that show both, reachability of the optimal solution and convergence of the density with probability one. In this sense, these sequences balance what is sometimes called exploitation and exploration. However, at present we are not able to show that the limiting density is concentrated on optimal solutions. Our result also shows the difference between the two types of convergence: convergence of the density does not necessarily mean that the sample process is absorbed after finitely many iterations.

Applied to ACOs, our results complement the convergence results of [7] and [8] to the case where the update of pheromones only uses current solutions. They also show that, with this update, ACOs will end in suboptimal solutions with a positive probability if a constant evaporation rate is used. See Section IV for details on ACOs.

The paper starts with an exact definition of the encoding of the solutions, the CE algorithm and the stochastic model describing its evolution in Section II. In Section III we state the main results and discuss their implications. In Section IV, we discuss some extensions of our model and how it applies to the ACO algorithms. The very complex and technical proofs together with some auxiliary results are collected in Section V. Section VI contains a conclusion and an outlook on further research.

II. The Mathematical Model

A. Problem Encoding

We are considering a problem of discrete optimization with a finite set of feasible *solutions* S and a *cost function* $f : S \to \mathbb{R}$. Let $S^* := \{s \in S \mid f(s) = \min_{s' \in S} f(s')\}$ be the set of *optimal solutions*. To exclude trivial cases, we assume that $|S| > 1$ and that $S^* \neq S$.

We assume further that S has a particular structure: each feasible solution $s \in S$ can be written as a finite string $s = (s_1, \ldots, s_L)$ of symbols from a finite alphabet $\mathcal{A} := \{a_1, \ldots, a_K\}$. L is the fixed length of the strings.

Often, CE models (e.g. in [3]) use an unconstrained solution space $S = \mathcal{A}^L$. We generalize this model introducing a feasibility function $C_i(y, a)$ that assigns a weight to each $a \in \mathcal{A}$ for every possible partial solution y of length i. Thus, if $C_i(y, a) = 0$, the partial solution y cannot be continued by adding the symbol a. The larger the value of $C_i(y, a)$, the more desirable it seems, from a greedy point of view, to continue with symbol a. We assume that these values are normed, an example is given after the formal definition below.

Our CE method, described in Subsection II-B below, constructs solution strings stepwise by adding new feasible symbols to the right end of a partial solution until it is complete. More formally, we define the set R_i of *feasible partial solutions* of length i recursively as follows: Let $\diamond$ denote the empty string over $\mathcal{A}$. We assume that we are given

$$C_0(\diamond, \cdot) : \mathcal{A} \to [0, 1], \qquad \sum_{a \in \mathcal{A}} C_0(\diamond, a) = 1,$$

expressing the feasibility or desirability of a symbol at the first position of a solution. We then define

$$R_1 := \{a \in \mathcal{A} \mid C_0(\diamond, a) > 0\}$$

as the set of feasible partial solutions of length 1. Assume now that we have defined R_i for some $i \in \{0, \ldots, L-1\}$ and that we are given

$$C_i(y, \cdot) : \mathcal{A} \to [0, 1], \quad \sum_{a \in \mathcal{A}} C_i(y, a) = 1 \quad \text{for each } y \in R_i.$$

Let (y, a) denote the concatenation of symbol $a \in \mathcal{A}$ to the right end of the partial solution (string of symbols) $y \in R_i$. Then we define

$$R_{i+1} := \big\{(y, a) \mid y \in R_i, a \in \mathcal{A}, C_i(y, a) > 0\big\}$$

and put $S := R_L$. For $y \in R_i, i \in \{0, \ldots, L-1\}$, let

$$C_i(y) := \{a \in \mathcal{A} \mid C_i(y, a) > 0\}$$

be the support of $C_i(y, \cdot)$. We use the abbreviation $I := \{0, \ldots, L-1\}$ in the sequel.

As an example, we look at a Traveling Salesman Problem (TSP), where the aim is to find a tour of minimal length through K cities $\{a_1, \ldots, a_K\}$. Here, $C_0(\diamond, a) = 0$ could be used to prevent tours from starting in city a. If $y \in R_i$ is a partial tour of length i, then $C_i(y)$ contains all cities that could be visited next. $C_i(y, a)$ would be zero if a has been visited before or cannot be reached from y. $0 < C_i(y, a)$ could be chosen as the normed distance from (the end of) y to a. Here, the distance has to be normed by the sum of distances from y to all $a' \in C_i(y)$.

This feasibility concept allows to include the case where there is only a set $C_i(y) \subset \mathcal{A}$ of feasible continuations of the partial solution $y \in R_i$ and no desirability information. In this case, $C_i(y, \cdot)$ is chosen to be the uniform distribution on $C_i(y)$. In particular, if there are no constraints at all, we have $C_i(\cdot) \equiv \mathcal{A}$ for all $i \in \{0, \ldots, L-1\}$ and put

$$C_i(y, a) \equiv \frac{1}{|\mathcal{A}|}, \qquad \text{for all } y \in R_i, a \in \mathcal{A}. \tag{II.1}$$

We shall refer to (II.1) as the 'unrestricted case'.

Although the construction of $S = R_L$ seems to be very particular, it imposes no restrictions on the optimization problems. Formally, we can include any finite set S of feasible

This is the author's version of an article that has been published in this journal. Changes were made to this version by the publisher prior to publication.
The final version of record is available at http://dx.doi.org/10.1109/TEVC.2014.2336882

solutions into our model by choosing $\mathcal{A} := S, L := 1$ and $C_0(\diamond, s) > 0$ for all $s \in S$. Thus we can force any problem into our framework, but the efficiency of the method (which is not discussed in this paper) may not be high. It is more reasonable to use our model in cases where $|\mathcal{A}| \ll |S|$.

B. The Generalized CE Algorithm

The cross entropy algorithm essentially evolves a distribution on the set $S = R_L$ of all feasible solutions with the aim to give a high probability to the optimal solutions from S^*.

Let $\mathbb{P}(\mathcal{A})$ denote the set of all probability measures on the set $\mathcal{A}$. Then $\boldsymbol{p} \in \mathbb{P}(\mathcal{A})^L$, $\boldsymbol{p} = (\boldsymbol{p}(1), \dots, \boldsymbol{p}(L))$ is a product probability measure on $\mathcal{A}^L$, that describes the selection of a solution $s = (s_1, \dots, s_L) \in \mathcal{A}^L$, where the L symbols $s_1, \dots, s_L$ are chosen independently of each other. Here, $\boldsymbol{p}(i) = \big(\boldsymbol{p}(a; i)\big)_{a \in \mathcal{A}} \in \mathbb{P}(\mathcal{A})$ is the distribution for the symbol on the i-th position of the string.

Our CE algorithm takes the following items as input:

- the feasibility distributions $C_i(\cdot, \cdot), i \in I$;
- a sequence of *smoothing parameters* $(\varrho_t)_{t \geq 1}$ with $\varrho_t \in (0, 1)$;
- a sample size $N \in \mathbb{N}$ and a subsample size $N_b \leq N$;
- a starting distribution $\boldsymbol{p}_0 \in \mathbb{P}(\mathcal{A})^L$.

Starting: For time-step $t = 0$, put $\boldsymbol{p} := \boldsymbol{p}_0$, then run iteratively through the following steps for $t = 1, 2, \dots$ until some stopping criterion is fulfilled.

Sampling: If the present distribution is $\boldsymbol{p} \in \mathbb{P}(\mathcal{A})^L$, a solution $s = (s_1, \dots, s_L) \in S$ is drawn according to the probability

$$Q_{\boldsymbol{p}}(s) := Q_{\boldsymbol{p}}(s_1; 1, \diamond) \cdot \prod_{i=2}^{L} Q_{\boldsymbol{p}}(s_i; i, (s_1, \dots, s_{i-1})), \tag{II.2}$$

where

$$Q_{\boldsymbol{p}}(a; i, y) := \frac{\boldsymbol{p}(a, i) C_{i-1}(y, a)}{\sum_{a' \in \mathcal{A}} \boldsymbol{p}(a', i) C_{i-1}(y, a')} \tag{II.3}$$

is the probability that the feasible symbol $a \in \mathcal{A}$ is added at position i to the feasible partial solution $y \in R_{i-1}$. We use the convention $\frac{0}{0} = 0$ throughout this paper.

In this way, the CE algorithm draws N solutions $s^{(1)}, \dots, s^{(N)}$ independently and identically distributed (i.i.d.).

Evaluation: This sample $\boldsymbol{x} := (s^{(1)}, \dots, s^{(N)})$ is ordered according to its cost values

$$f(s^{(n_1)}) \leq f(s^{(n_2)}) \leq \dots \leq f(s^{(n_N)})$$

and the better N_b solutions are selected: $\mathcal{N}_b := \{s^{(n_1)}, s^{(n_2)}, \dots, s^{(n_{N_b})}\}$. Then we define the relative frequency of symbol a at position $i \in \{1, \dots, L\}$ in the selected part of the sample:

$$w(a; i, \boldsymbol{x}) := \frac{1}{N_b} \sum_{s \in \mathcal{N}_b} \mathbb{1}_{\{a\}}(s_i) \tag{II.4}$$

and collect these frequencies for all $a \in \mathcal{A}$ to form $w(i, \boldsymbol{x}) := \big(w(a; i, \boldsymbol{x})\big)_{a \in \mathcal{A}}$ and

$$\boldsymbol{w}(\boldsymbol{x}) := \big(w(1, \boldsymbol{x}), \dots, w(L, \boldsymbol{x})\big). \tag{II.5}$$

Then $\boldsymbol{w}(\boldsymbol{x})$ is a product probability measure from $\mathbb{P}(\mathcal{A})^L$, that gives the relative frequencies of symbols in the better part of the sample $\boldsymbol{x}$ drawn with $Q_{\boldsymbol{p}}$.

Update: We update the present distribution $\boldsymbol{p}$ as a convex combination of $\boldsymbol{p}$ and the relative frequencies $\boldsymbol{w}(\boldsymbol{x})$:

$$\boldsymbol{p} := (1 - \varrho_{t+1})\boldsymbol{p} + \varrho_{t+1}\boldsymbol{w}(\boldsymbol{x}). \tag{II.6}$$

Next, the counter t is increased by 1 and the step 'Sampling' is performed with the new $\boldsymbol{p}$.

Some extensions to this model and a comparison to other approaches can be found in Section IV below.

C. The Solution Process

Applying the above algorithm iteratively results in a sequence of probability measures and samples that form a stochastic process

$$\Big(\boldsymbol{\Pi}_t; \boldsymbol{X}_t\Big)_{t=0,1,\dots}$$

where $\boldsymbol{\Pi}_t = \big(\Pi_t(1), \dots, \Pi_t(L)\big)$ is a random variable taking on values in $\mathbb{P}(\mathcal{A})^L$, that describes the distribution underlying the sampling in the t-th iteration. $\boldsymbol{\Pi}_t$ is called the *density* of the algorithm with $\boldsymbol{\Pi}_0 = \boldsymbol{p}_0$. $\boldsymbol{X}_t$ takes on values in S^N and is the sample of N solutions produced in the t-th 'Sampling' step of the algorithm, using $Q_{\boldsymbol{\Pi}_t}$ as defined in (II.2). In the sequel, we put $\boldsymbol{X}_t = \big(\boldsymbol{X}_t^{(1)}, \dots, \boldsymbol{X}_t^{(N)}\big)$ and $\boldsymbol{X}_t^{(n)} = \big(\boldsymbol{X}_t^{(n)}(1), \dots, \boldsymbol{X}_t^{(n)}(L)\big)$. Then $\boldsymbol{X}_t^{(n)}(i)$ denotes the symbol at position $i, 1 \leq i \leq L$, drawn in the n-th solution in iteration t. Similarly, $\boldsymbol{X}_t^{(n)}(1, \dots, i)$ denotes the partial solution up to the i-th position in that solution.

The process $(\boldsymbol{\Pi}_t, \boldsymbol{X}_t)_{t \geq 0}$ is well defined on a joint probability space $\big(\Omega, \mathbf{P}\big)$. It can be shown, using (II.2) and (II.6), that $\big(\boldsymbol{\Pi}_t,\ \boldsymbol{X}_t\big)_{t \geq 0}$ as well as the marginal process $\big(\boldsymbol{\Pi}_t\big)_{t \geq 0}$ are Markov processes.

Due to the deterministic nature of the update mechanism, we may also write $\boldsymbol{\Pi}_{t+1} = (1 - \varrho_{t+1})\boldsymbol{\Pi}_t + \varrho_{t+1}\boldsymbol{w}(\boldsymbol{X}_t)$ on a vector level and on the more detailed level

$$\begin{aligned} \boldsymbol{\Pi}_{t+1}(a;\ i) \\ = (1 - \varrho_{t+1})\boldsymbol{\Pi}_t(a;\ i) + \varrho_{t+1} w(a;\ i, \boldsymbol{X}_t) \end{aligned} \tag{II.7}$$

for each $a \in \mathcal{A}, i = 1, \dots, L$. We shall refer to (II.7) as the 'basic recursion', most of our results are based on it.

D. General assumptions

Throughout this paper we assume that the starting distribution $\boldsymbol{\Pi}_0 = \boldsymbol{p}_0$ is given such that any item from $\mathcal{A}$ has a positive probability at any position i :

$$\boldsymbol{p}_0(a; i) > 0 \qquad \text{for all } a \in \mathcal{A}, i = 1, \dots, L. \tag{II.8}$$

Also, without loss in generality we may assume that there are at least two solutions

$$s = (s_1, \dots, s_L), s' = (s'_1, \dots, s'_L) \ \text{ with } \ s_1 \neq s'_1. \tag{II.9}$$

Otherwise, all solutions would start with the same symbol, which could then be dropped from the encoding of the solutions.

III. Main results

In this Section we collect the main results of this paper and discuss their meaning. The quite complicated proofs are given in Section V.

A. Results on the Reachability of Optimal Solutions

Let $\mathcal{X}_t := \{\boldsymbol{X}_t^{(1)}, \dots, \boldsymbol{X}_t^{(N)}\}$ be the set of solutions sampled in the t-th iteration, then

$$\tau := \min\{t \geq 0 \mid \mathcal{X}_t \cap S^* \neq \emptyset\} \tag{III.1}$$

denotes the first iteration, in which an optimal solution is sampled. We now investigate whether we can guarantee to reach an optimal solution in finite time, i.e. whether $\mathbf{P}(\tau < \infty) = 1$.

Theorem 1 shows, that the results of [3] hold in our more general model.

Theorem 1. **a)** If $\sum_{t=1}^{\infty} \prod_{m=1}^{t} (1 - \varrho_m)^L = \infty$, then $\mathbf{P}(\tau < \infty) = 1$.
b) If $\mathbf{P}(\tau < \infty) = 1$, then $\sum_{t=1}^{\infty} \prod_{m=1}^{t} (1 - \varrho_m) = \infty$.
c) If $\varrho_t \equiv \varrho > 0$ is a constant, then we have $\mathbf{P}(\tau < \infty) < 1$, but $\mathbf{P}(\tau < \infty) \to 1$, if either $N \to \infty$ or $\varrho \to 0$.

The proof of this Theorem along the lines of [3] is given in Section V-B below. Theorem 1 b) shows in particular that for a constant smoothing parameter, i.e. $\varrho_t \equiv \varrho$, it cannot be guaranteed that the optimal solution is reached in finite time, as we have $\sum_{t=1}^{\infty} \prod_{m=1}^{t} (1 - \varrho) = 1/\varrho - 1 < \infty$. But then also the expected runtime, i.e. the expected number of solutions sampled before an optimal solution is reached, will be infinite for arbitrary optimization problems.

For a particular problem however, namely the LeadingOne problem, see e.g. [4], we are able to show that the runtime is bounded by a polynomial in the problem size with a probability converging to 1, even if ϱ_t is constant. In the LeadingOne problem we consider the unrestricted case with $\mathcal{A} = \{0, 1\}, S = \{0, 1\}^L$ and

$$f(s) := L - \sum_{l=1}^{L} \prod_{i=1}^{l} s_i \quad \text{for } s = (s_1, \dots, s_L). \tag{III.2}$$

Minimizing $f(s)$ then means to maximize the number of consecutive 1s counted from the left of the solution.

Theorem 2 extends the runtime results of [4], who consider a special case of the CE algorithm with $\varrho_t \equiv 1$ on a LeadingOne problem. The Theorem essentially states that if we take $\varrho_t \equiv \varrho \in (0, 1)$ and let the sample size grow as $N = L^{2+\epsilon}$ for some $\epsilon > 0$, then we can reach the optimal solution in L iterations with a probability converging to 1, i.e. the runtime is $O(L^{3+\epsilon})$ in a stochastic sense.

Theorem 2. Let $\varrho_t \equiv \varrho$, sample size $N = L^{2+\epsilon}$ for some $\epsilon > 0$ and $N_b = \lfloor \beta N \rfloor$ for some $0 < \beta < \frac{1}{3e} \prod_{m=1}^{\infty} \left(1 - (1-\varrho)^m\right)$. Let $\boldsymbol{\Pi}_0(1, i) \equiv \frac{1}{2}$, i.e. we start with the uniform distribution. Then for a LeadingOne problem, defined in (III.2), we have $\mathbf{P}(\tau < L) \to 1$ as $L \to \infty$.

For a proof of Theorem 2, see Subsection V-C below.

Theorem 1 b) and the lower bound in Lemma 10 b) below show that a constant smoothing parameter reduces the global 'exploration' of the search space. Theorem 2 however shows that we can compensate for that by increasing the local 'exploitation' in terms of a growing sample size, at least in specific problems.

B. Results on Convergence of Density and Samples

An important question, also addressed in [3], is whether the density $\boldsymbol{\Pi}_t$ will converge to a density concentrated on one point, and whether this point is an optimal solution. For the unrestricted case, [3] showed that the algorithm with $\varrho_t \equiv \varrho$ has a convergent density in the following sense:

Definition 3. We say that the algorithm has *convergent* densities, if $\left(\boldsymbol{\Pi}_t\right)_{t \geq 0}$ converges almost surely against a product of one-point measures, i.e.

$$\mathbf{P}\Big(\forall i = 1, \dots, L \quad \forall a \in \mathcal{A} \quad \lim_{t \to \infty} \boldsymbol{\Pi}_t(a\,;\, i) \in \{0, 1\}\Big) = 1.$$

A convergent density means that asymptotically, we only sample one identical solution. But this may also happen after finitely many iterations:

Definition 4. We say that the algorithm has *convergent samples*, if $\left(\boldsymbol{X}_t\right)_{t \geq 0}$ converges against some fixed sample $\boldsymbol{x} = (s, \dots, s) \in S^N$ of identical solutions, almost surely, i.e. if the following event has probability one:

$$\exists s \in S \; \exists T \in \mathbb{N} \; \forall m \geq 1 \; \forall n = 1, \dots, N \quad X_{T+m}^{(n)} = s.$$

In this case, the sampling freezes to a single solution after the finite random time T and no more progress is possible after that time. Theorem 5 shows that this does happen, if a constant smoothing parameter is used.

Theorem 5. **a)** If $\varrho_t \geq \varrho$ for some $\varrho > 0$, then the algorithm has convergent samples.
b) For $|S^*| = 1$, convergent samples imply $\mathbf{P}(\tau < \infty) < 1$, hence convergence of samples and $\mathbf{P}(\tau < \infty) = 1$ are mutually exclusive in this case.
c) Convergent samples imply a convergent density.
d) If the density converges, then $\sum_{t=1}^{\infty} \varrho_t = \infty$.

The quite complex proof of Theorem 5 is given in Section V-D below. Theorem 5 a) shows that the phenomenon of 'genetic drift' may also occur in this generalized CE provided a constant smoothing parameter is employed. From Theorem 5 b), we see that in this case an optimal solution may not be found if it is unique.

C. Reachability and Convergent Density are Compatible

In [3], the question was raised whether convergence of densities and reachability of optimal solutions are mutually exclusive, and this remained an open question. In Theorem 5, a partial answer was given with respect to the stronger concept of convergent samples.

At least for the unrestricted case, we can give a complete answer: there are smoothing sequences such that the algorithm has $\mathbf{P}(\tau < \infty) = 1$, it has convergent densities but no convergent samples. This also shows that the opposite direction in

Theorem 5 c) does not hold in general, densities may converge without freezing the samples in finite time.

Theorem 6. In the unrestricted case the following holds: Assume that $\varrho \in (0,1)$ and $c_k \in \mathbb{N}$ for $k = 0, 1, \ldots,$ with $c_0 = 1$, are chosen such that

$$\sum_{k=1}^{\infty} c_k (1-\varrho)^{kL} = \infty.$$

Define $e_k := \sum_{i=1}^{k} c_{i-1},\ k \geq 1$, and let $x_k \in (0,1)$ be any sequence such that $\sum_{k=1}^{\infty} x_k < \infty$. We may now define a smoothing sequence

$$\varrho_t := \begin{cases} \varrho & \text{if } t = e_k \text{ for some } k \geq 1 \\ 1-(1-x_k)^{\frac{1}{c_k - 1}} & \text{if } e_k < t < e_{k+1} \text{ for some } k \geq 1 \end{cases} \tag{III.3}$$

for $t \geq 1$. Then the algorithm has convergent densities, it has $\mathbf{P}(\tau < \infty) = 1$, and its samples do not converge.

The proof of Theorem 6 is again given in Section V-E.

As an example for the values in Theorem 6, take an arbitrary $\varrho > 0$, then one may choose $c_k = (1-\varrho)^{-kL}$ to obtain

$$\sum_{k=1}^{\infty} c_k (1-\varrho)^{kL} = \sum_{k=1}^{\infty} 1 = \infty.$$

For $\varrho < 1/2$, one could, for example, use $c_k := 2^{kL}$.

Theorem 6 shows that a global exploration of the search space (that implies reachability) and the ability for thorough local search, that includes some kind of convergence, can be balanced in this type of algorithm. It also shows that the two types of convergence are different, convergence in the continuous space of densities does not imply convergence in the finite space of samples. We must admit, however, that we are currently not able to give a smoothing sequence under which the limiting density is concentrated on some optimal solution.

IV. Extensions and Comparison to ACOs

The evaluation step in (II.4) simply calculates the relative frequencies in the better part of the sample. A thorough inspection of the proofs below shows that the results of the above Theorems (except Theorem 2) still hold, if we replace the relative frequencies $w(a; i, \boldsymbol{x})$ by some other measure $0 \leq \tilde{w}(a; i, \boldsymbol{x}) \leq 1$ not depending on t with the following properties:

(i) $\sum_{a \in \mathcal{A}} \tilde{w}(a; i, \boldsymbol{x}) = 1$, i.e. $\tilde{w}(\cdot; i, \boldsymbol{x})$ is a probability measure on $\mathcal{A}$;

(ii) $\tilde{w}(a; i, \boldsymbol{x}) = 1$ if all solutions in the sample $\boldsymbol{x}$ have symbol a at their $i-$th position;

(iii) $\tilde{w}(a; i, \boldsymbol{x}) = 0$ if none of the solutions in the sample $\boldsymbol{x}$ has symbol a at its $i-$th position.

For example, $\tilde{w}$ could be the relative frequencies of a randomly selected subset $\mathcal{N}$ of $\boldsymbol{x}$, or the weighted relative frequencies where the weights reflect the cost $f(s)$ of the solution as in

$$\tilde{w}(a; i, \boldsymbol{x}) := \frac{\sum_{s \in \mathcal{N}} \mathbb{1}_{\{a\}}(s_i) g(f(s))}{\sum_{s \in \mathcal{N}} g(f(s))}, \tag{IV.1}$$

here g is some decreasing function. $\tilde{w}$ may even include some kind of memory, as long as properties *(i)-(iii)* hold. In the proofs of Section V below, we will stick to the original frequency model as used in [3].

With these extensions, our model covers many heuristic optimization algorithms of the model-based type. Therefore the results given above will apply to all these algorithms.

In particular, they apply to ACO algorithms as described in [9] and [10]. Here solutions are maximal loop-free paths in a so-called construction graph. The arcs (k, l) of the path are selected according to 'pheromones' τ_{kl} and a 'visibility' η_{kl} on the arcs. Taking arcs as symbols $a := (k, l)$, pheromones can be expressed by the density in our model, and the visibility together with the loop-freeness can go into our feasibility measure $C_i(y, a)$. The evaporation rate for the update of pheromones is identical to the smoothing parameter, hence solutions in our CE model are sampled with the same probability as they are constructed by ants, if we choose (IV.1) as evaluation of the solutions observed. Note that our model requires density and feasibility to be normed probability measures, but this does not affect the sampling probabilities as given, for example, by [9].

Convergence results of ACOs can be found in [10], [7] and [8]. They concentrate on ACOs with an update of pheromones that only uses the best solution found so far and restricted pheromone values in [7], whereas our model has no restrictions on the density and uses the present sample for update.

With these differences, our results on reachability complement those found in [10] and [8]. In particular, the assumptions made in [7] fulfill the sufficient condition of reachability in Theorem 1 a). The results on absorption for constant evaporation rate and the possible balancing in Theorems 5 and 6 also carry over to ACOs. Our runtime result in Theorem 2 complements findings in [11], [12] and [13], who inspect the runtime of ACOs with restricted pheromone values (max-min ant system) on the so-called OneMax problem.

Using best-so-far update, as in the ACO models cited above, introduces a monotonicity that allows to show convergence to an optimal solution once it has been sampled, see [7]. Currently, we are not able to show such a result for our algorithm.

V. Proofs of the Theorems

A. Some Auxiliary Results

We start with the basic recursion in (II.7), which is crucial for our work. For a fixed pair $(a; i)$, this recursion fulfills the condition (V.1) of the following Lemma 7, and therefore the conclusions (V.2) - (V.4) hold. This will be used throughout this paper. We assume as usual $\prod_{i=m}^{k} \cdots \equiv 1$ for $m > k$.

Lemma 7. Let $0 < r_t < 1$ for $t = 1, 2, \ldots$.

a) $$\sum_{t=1}^{\infty} r_t = \infty \iff \prod_{t=1}^{\infty} (1 - r_t) = 0$$
$$\iff \prod_{t=1}^{\infty} (1 - c r_t) = 0 \qquad \text{for any } 0 < c < 1.$$

b) $\sum_{m=1}^{t} r_m \prod_{i=m+1}^{t}(1-r_i) \;=\; 1-\prod_{m=1}^{t}(1-r_m)$ for any $t \geq 1$.

c) For a given sequence $w_t \in [0,1], t = 0,1,\ldots,$ and $q_0 \in (0,1)$, the recursion

$$q_{t+1} = (1-r_{t+1})q_t + r_{t+1}w_t, \quad t \geq 0, \tag{V.1}$$

has the unique solution

$$q_t = q_0 \prod_{m=1}^{t}(1-r_m) + \sum_{m=1}^{t} r_m w_{m-1} \prod_{i=m+1}^{t}(1-r_i) \tag{V.2}$$

with

$$0 < q_0 \prod_{m=1}^{t}(1-r_m) \;\leq\; q_t \tag{V.3}$$

$$\leq\; 1-(1-q_0)\prod_{m=1}^{t}(1-r_m) < 1, \quad t \geq 0.$$

If, in particular, $w_m \equiv w \in [0,1]$ for $m = 0,\ldots,t-1$, then

$$q_t = w - (w-q_0)\cdot \prod_{m=1}^{t}(1-r_m). \tag{V.4}$$

Proof: (see [3]) **a)** The first part is a standard result, the second follows as $\sum_{i=0}^{\infty} r_i = \infty \iff \sum_{i=0}^{\infty} cr_i = \infty$. **b)** follows, if $r_m = 1-(1-r_m)$ is used on the left hand side. **c)** (V.2) can be proven using induction on t. (V.4) follows from (V.2) using b). ■

Now, (V.3) applied to (II.7) and (II.8) shows that

$$0 < \mathbf{\Pi}_t(a;i) < 1 \tag{V.5}$$

for all $t \geq 0, i \in \{1,\ldots,L\}$ and $a \in \mathcal{A}$. We then also have

$$0 < C_i(y,a)\mathbf{\Pi}_t(a;i{+}1) < 1 \tag{V.6}$$

for all $t \geq 0, i \in I, y \in R_i$ and $a \in C_i(y)$.

In the unrestricted case as in [3], we have $Q_{\mathbf{\Pi}_t}(a;i{+}1,y) = \mathbf{\Pi}_t(a;i{+}1)$, i.e. the probability to continue a feasible partial solution $y \in R_i$ with a symbol a in the t-th iteration coincides with the density $\mathbf{\Pi}(a;i{+}1)$. We may therefore directly use the basic recursion (II.7) to derive the asymptotic behavior of the procedure from Lemma 7. In the general constrained case this is not possible, as $Q_{\mathbf{\Pi}_t}$ does not fulfill an equation as (II.7). But we can find a bounding probability $Q'_{\mathbf{\Pi}_t}$ for $Q_{\mathbf{\Pi}_t}$, for which a similar recursion holds under certain conditions (see Lemma 11 below).

For $\boldsymbol{p} \in \mathbb{P}(\mathcal{A})^L, i \in I, y \in R_i$ and $a \in \mathcal{A}$ define

$$Q'_{\boldsymbol{p}}(a;i{+}1,y) := \begin{cases} \frac{\boldsymbol{p}(a;i+1)}{\sum_{a' \in C_i(y)} \boldsymbol{p}(a';i+1)} & \text{if } a \in C_i(y), \\ 0 & \text{otherwise,} \end{cases} \tag{V.7}$$

with $\frac{0}{0} = 0$. This is similar to $Q_{\boldsymbol{p}}$ except that the feasibility distribution $C_i(y,a)$ has been replaced by a constant on its support $C_i(y)$. Note, that for the unrestricted case of (II.1), $Q_{\boldsymbol{p}}$, $Q'_{\boldsymbol{p}}$ and $\boldsymbol{p}$ all coincide.

To bound $Q_{\boldsymbol{p}}(a;i{+}1,y)$ with the help of $Q'_{\boldsymbol{p}}(a;i{+}1,y)$, we first need bounds on $C_i(y,a)$. Let

$$\eta := \max\big\{C_i(y,a) \mid y \in R_i,\; a \in \mathcal{A},\; i \in I\big\} \tag{V.8}$$

and

$$\lambda := \min\big\{C_i(y,a) > 0 \mid y \in R_i, a \in \mathcal{A}, i \in I\big\}. \tag{V.9}$$

Obviously, $0 < \lambda \leq \eta \leq 1$. Now define the following two bounding functions for $x \in [0,1]$

$$h(x) := \frac{\eta x}{\lambda + (\eta-\lambda)x}, \quad \ell(x) := \frac{\lambda x}{\eta - (\eta-\lambda)x}. \tag{V.10}$$

Then $\ell(x) \leq x \leq h(x)$ and in the unrestricted case $h(x) = \ell(x) = x$. Lemma 8 collects some properties of h and ℓ.

Lemma 8. Let h, ℓ be as defined in (V.10).

a) h and ℓ are strictly increasing and continuous taking values in $[0,1]$. h is concave, ℓ is convex with $\ell(x) \leq x \leq h(x), x \in [0,1]$.

b) For $\eta = \lambda$ we have $h(x) = \ell(x) = x$ for $x \in [0,1]$, for $\eta > \lambda$ we have $h(x) = x = \ell(x) \iff x \in \{0,1\}$.

c) $h(x) = 1-\ell(1-x)$ and $\ell(x) = 1-h(1-x)$.

d) Let $0 \leq x_n \leq 1, n \in \mathbb{N}$, be a convergent sequence, then for any $0 < c < 1$, the following series are either all convergent or all divergent

$$\sum_{n=1}^{\infty} x_n, \quad \sum_{n=1}^{\infty} h(x_n), \quad \sum_{n=1}^{\infty} \ell(x_n),$$
$$\sum_{n=1}^{\infty} h(cx_n), \quad \sum_{n=1}^{\infty} \ell(cx_n)$$

Proof: Assertions a) - c) are straightforward to show. Assertion d) follows immediately from the well-known limit comparison test. ■

Lemma 9 below collects some properties of $Q_{\mathbf{\Pi}_t}$ that are needed throughout the paper. Recall that $\diamond$ is the empty string that we use as a starting value for the recursions. Note that some assertions only hold if $|C_i(y)| > 1$. This excludes the case that there is just one possible continuation of y, which then must have probability one.

Lemma 9. Let $i \in I, y \in R_i$ and $a \in C_i(y)$. Then the following holds.

a) $\ell\big(Q'_{\mathbf{\Pi}_t}(a;i+1,y)\big) \leq\; Q_{\mathbf{\Pi}_t}(a;i+1,y) \;\leq\; h\big(Q'_{\mathbf{\Pi}_t}(a;i+1,y)\big)$ for all $t \in \mathbb{N}$.

b) If $|C_i(y)| = 1$ then $Q_{\mathbf{\Pi}_t}(a;i+1,y) = Q'_{\mathbf{\Pi}_t}(a;i+1,y) = 1$.
If $|C_i(y)| > 1$, then $0 < Q'_{\mathbf{\Pi}_t}(a;i+1,y) < 1$ and $0 < Q_{\mathbf{\Pi}_t}(a;i+1,y) < 1$ for any $t \geq 0$.

Proof: **a)** In view of Lemma 8 b) we only have to consider the case $|C_i(y)| > 1$.

$$\begin{aligned} Q_{\boldsymbol{p}}(a;i+1,y) &= \frac{\mathbf{\Pi}_t(a;i+1)C_i(y,a)}{\sum_{a' \in \mathcal{A}} \mathbf{\Pi}_t(a';i+1)C_i(y,a')} \\ &\leq \frac{\eta \mathbf{\Pi}_t(a;i+1)}{\eta \mathbf{\Pi}_t(a;i+1) + \lambda \sum_{a' \in C_i(y), a \neq a'} \mathbf{\Pi}_t(a';i+1)} \\ &= h\big(Q'_{\mathbf{\Pi}_t}(a;i+1,y)\big), \end{aligned}$$

where we use (V.5). Similarly, the left-hand inequality of a) is derived.

b) From (V.5) we have $Q'_{\mathbf{\Pi}_t}(a;i{+}1,y) > 0$. If $|C_i(y)| > 1$, we see that $Q'_{\mathbf{\Pi}_t}(a;i{+}1,y) < 1$ for any $a \in C_i(y)$ and $t \in \mathbb{N}$, this implies the conclusion using Lemma 8 b) and part a). ■

B. Proof of Theorem 1

We first give lower bounds on the probabilities of a) reaching the set S^* of optimal solutions and b) of staying forever with one (possibly non-optimal) solution .

Lemma 10. **a)** For $s = (s_1, \dots, s_L) \in S$ and $\boldsymbol{p}_0 \in \mathbb{P}(\mathcal{A})^L$ let

$$c(s) := C_0(\diamond, s_1) \prod_{i=2}^{L} C_{i-1}\big((s_1, \dots, s_{i-1}), s_i\big) \text{ and}$$

$$\delta(s) := \boldsymbol{p}_0(s) \cdot c(s).$$

Then for τ as defined in (III.1)

$$\mathbf{P}(\tau < \infty) \tag{V.11}$$

$$\geq 1 - \min_{s^* \in S^*} \Big[\prod_{t=0}^{\infty} \Big(1 - \delta(s^*) \prod_{m=1}^{t} (1 - \varrho_m)^L \Big) \Big]^N.$$

b) Let $s \in S$, then, with h as defined in (V.10),

$$\mathbf{P}(\boldsymbol{X}_t^{(n)} = s \text{ for } n = 1, \dots, N \text{ and } t = 0, 1, \dots)$$

$$\geq Q_{\boldsymbol{\Pi}_0}(s)^N \Big[\prod_{t=1}^{\infty} \Big(1 - h\big(\prod_{m=1}^{t} (1 - \varrho_m)\big)\Big) \Big]^{LN}.$$

Note that the bound in part a) holds for the first hitting time of any subset of solutions, so it is in fact a bound for the probability that arbitrary solutions will be visited in finite time.

Proof of Lemma 10: **a)** (cf. [3]) We fix an arbitrary optimal solution $s^* = (s_1^*, \dots, s_L^*) \in S^*$, then we have

$$\mathbf{P}(\tau = \infty) = \mathbf{P}\Big(\bigcap_{t=0}^{\infty} [S^* \cap \mathcal{X}_t = \emptyset]\Big) \leq \mathbf{P}\Big(\bigcap_{t=0}^{\infty} [s^* \notin \mathcal{X}_t]\Big)$$

$$= \mathbf{P}(s^* \notin \mathcal{X}_0) \prod_{t=1}^{\infty} \mathbf{P}\big[s^* \notin \mathcal{X}_t \mid s^* \notin \mathcal{X}_m, m = 0, \dots, t-1\big]. \tag{V.12}$$

We now derive an upper bound for the factors in (V.12). First we have

$$\mathbf{P}\big[s^* \notin \mathcal{X}_t \mid s^* \notin \mathcal{X}_m, m = 0, \dots, t-1\big] \tag{V.13}$$

$$= \mathbf{E}\Big[\mathbf{P}\big[s^* \notin \mathcal{X}_t \mid \boldsymbol{\Pi}_t\big] \;\Big|\; s^* \notin \mathcal{X}_m, m = 0, \dots, t-1\Big].$$

From the definition (II.3) of $Q_{\boldsymbol{p}}$ and (V.3) of Lemma 7 applied with $q_0 = \boldsymbol{\Pi}_0(a; i)$ we have

$$Q_{\boldsymbol{\Pi}_t}(a; i, y) \geq \boldsymbol{\Pi}_t(a; i) C_{i-1}(y, a)$$

$$\geq \boldsymbol{\Pi}_0(a; i) C_{i-1}(y, a) \prod_{m=1}^{t} (1 - \varrho_m)$$

for all $a \in A, y \in R_{i-1}$, $i \in \{1, \dots, L\}$ and $t \geq 0$. As the solutions are sampled i.i.d., we may now conclude

$$\mathbf{P}\big[s^* \notin \mathcal{X}_t \mid \boldsymbol{\Pi}_t\big] \tag{V.14}$$

$$= \Big(\mathbf{P}\big[s^* \neq X_t^{(1)} \mid \boldsymbol{\Pi}_t\big]\Big)^N = \Big(1 - \mathbf{P}\big[s^* = X_t^{(1)} \mid \boldsymbol{\Pi}_t\big]\Big)^N$$

$$\leq \Big(1 - \delta(s^*) \prod_{m=1}^{t} (1 - \varrho_m)^L\Big)^N.$$

For the first factor in (V.12), we have from (V.14) for $t = 0$

$$\mathbf{P}(s^* \notin \mathcal{X}_0) \leq \Big(1 - \delta(s^*)\Big)^N.$$

Combining these results, we obtain

$$\mathbf{P}(\tau < \infty) = 1 - \mathbf{P}(\tau = \infty)$$

$$\geq 1 - \Big[\prod_{t=0}^{\infty} \Big(1 - \delta(s^*) \prod_{m=1}^{t} (1 - \varrho_t)^L \Big) \Big]^N \tag{V.15}$$

As this holds for all $s^* \in S^*$, the assertion follows.

b) Let $s \in S$. We write $\boldsymbol{X}_t^{(\cdot)} \equiv s$ for $\boldsymbol{X}_t^{(n)} = s, n = 1, \dots, N$ and $\mathfrak{S}$ for the event $\boldsymbol{X}_m^{(\cdot)} \equiv s$ for all $m = 0, \dots, t-1$. Then, as samples are i.i.d.,

$$\mathbf{P}\big[\boldsymbol{X}_t^{(\cdot)} \equiv s, t = 0, 1, \dots\big] = \mathbf{P}\big[\boldsymbol{X}_0^{(\cdot)} \equiv s\big] \cdot \tag{V.16}$$

$$\cdot \prod_{t=1}^{\infty} \mathbf{P}\big[\boldsymbol{X}_t^{(\cdot)} \equiv s \mid \boldsymbol{X}_m^{(\cdot)} \equiv s \text{ for } m = 0, \dots, t-1\big]$$

$$= \mathbf{P}\big[\boldsymbol{X}_0^{(1)} = s\big]^N \cdot \prod_{t=1}^{\infty} \mathbf{P}\big[\boldsymbol{X}_t^{(1)} = s \mid \mathfrak{S}\ \big]^N.$$

We have $\mathbf{P}(\boldsymbol{X}_0^{(1)} = s) = Q_{\boldsymbol{\Pi}_0}(s)$ for the first factor. Using Lemma 9 a), we obtain for the other factors in (V.16) with $s_{|i} := (s_1, \dots, s_i)$

$$\mathbf{P}\big[\boldsymbol{X}_t^{(1)} = s \mid \mathfrak{S}\ \big] = \mathbf{P}\big[\boldsymbol{X}_t^{(1)}(1) = s_1 \mid \mathfrak{S}\ \big]$$

$$\cdot \prod_{i=2}^{L} \mathbf{P}\big[\boldsymbol{X}_t^{(1)}(i) = s_i \mid \boldsymbol{X}_t^{(1)}(1, \dots, i-1) = s_{|i-1},\ \mathfrak{S}\ \big]$$

$$= \mathbf{E}\big[Q_{\boldsymbol{\Pi}_t}(s_1; 1, \diamond) \mid \mathfrak{S}\ \big]$$

$$\cdot \prod_{i=2}^{L} \mathbf{E}\big[Q_{\boldsymbol{\Pi}_t}(s_i; i, s_{|i-1}) \mid \boldsymbol{X}_t^{(1)}(1, \dots, i-1) = s_{|i-1},\ \mathfrak{S}\ \big]$$

$$\geq \prod_{i=1}^{L} \mathbf{E}\big[\ell\big(Q'_{\boldsymbol{\Pi}_t}(s_i; i, s_{|i-1})\big) \big|\ \mathfrak{S}\ \big],$$

with $s_{|0} = \diamond$. Under the condition $\mathfrak{S}$, the relative frequencies $w(s_i; i, \boldsymbol{X}_m)$ are all equal to 1 for all $m = 0, \dots, t-1$, hence we may use Lemma 11 d) below (see the remark following Lemma 11 and (V.36)) to obtain

$$Q'_{\boldsymbol{\Pi}_t}(s_i; i, s_{|i-1}) \geq 1 - \prod_{m=1}^{t} (1 - \varrho_m)$$

for $t \geq 1$ and $i = 1, \dots, L$. Now, from (V.16), we obtain using Lemma 8 c)

$$\mathbf{P}\big[\boldsymbol{X}_t^{(\cdot)} \equiv s, t = 0, 1, \dots\big] \tag{V.17}$$

$$\geq Q_{\boldsymbol{\Pi}_0}(s)^N \Big[\prod_{t=1}^{\infty} \Big(1 - h\big(\prod_{m=1}^{t} (1 - \varrho_m)\big)\Big) \Big]^{LN}.$$

■

Proof of Theorem 1: **a)** From Lemma 7 a), we see that with $r_t := \prod_{m=1}^{t} (1 - \varrho_m)^L$ and $c := \delta(s^*)$, the assumption $\sum_{t=1}^{\infty} \prod_{m=1}^{t} (1 - \varrho_m)^L = \infty$ implies

$$\prod_{t=1}^{\infty} \Big(1 - \delta(s^*) \prod_{m=1}^{t} (1 - \varrho_m)^L\Big) = 0.$$

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The final version of record is available at http://dx.doi.org/10.1109/TEVC.2014.2336882

Now $\mathbf{P}(\tau < \infty) = 1$ follows from Lemma 10.

b) Let $s \in S - S^*$ with $Q_{\mathbf{\Pi}_0}(s) > 0$. Note that such an s must exist, otherwise we could only sample optimal solutions. If $\mathbf{P}(\tau < \infty) = 1$, the samples cannot be identical to s for all times, hence from Lemma 10 b) we must have

$$0 = \mathbf{P}(\boldsymbol{X}_t^{(\cdot)} \equiv s, t = 0, 1, \dots) \\ \geq Q_{\mathbf{\Pi}_0}(s)^N \Big[\prod_{t=1}^{\infty} \Big(1 - h\big(\prod_{m=1}^{t}(1 - \varrho_m)\big)\Big)\Big]^{LN},$$

and $\prod_{t=1}^{\infty}\Big(1 - h\big(\prod_{m=1}^{t}(1 - \varrho_m)\big)\Big) = 0$ must hold. From Lemma 7 a) we see that this implies

$$\sum_{t=1}^{\infty} h\Big(\prod_{m=1}^{t}(1 - \varrho_m)\Big) = \infty.$$

The assertion now follows from Lemma 8 d).

c) Assume $\varrho_t \equiv \varrho > 0$, we have $\sum_{t=1}^{\infty}\prod_{m=1}^{t}(1 - \varrho) < \infty$, hence by part b) of this Theorem, $\mathbf{P}(\tau < \infty) < 1$. Note that the infinite product in (V.15) in this case is smaller than 1 and continuous in ϱ, so (V.15) approaches 1 for either $N \to \infty$ or $\varrho \to 0$. ∎

C. *Proof of Theorem 2*

Proof: With $\mathcal{A} = \{0, 1\}$, we write $\pi_t(i) := \mathbf{\Pi}_t(1; i)$. As we consider the unrestricted case here, the sample $\boldsymbol{X}_t = (\boldsymbol{X}_t^{(1)}, \dots, \boldsymbol{X}_t^{(N)})$ can be viewed as a random matrix with mutually independent entries and solution $\boldsymbol{X}_t^{(n)}$ as n-th row. We denote by $\boldsymbol{X}_t^{[\cdot]} = \big(\boldsymbol{X}_t^{[1]}, \dots, \boldsymbol{X}_t^{[N]}\big)$ this matrix with rows ordered according to increasing cost function values : $f(\boldsymbol{X}_t^{[1]}) \leq \dots \leq f(\boldsymbol{X}_t^{[N]})$.

Then $\mathcal{N}_b$ is the sub-matrix of the first N_b rows of $\boldsymbol{X}^{[\cdot]}$ and the empirical distributions from $\mathcal{N}_b$ may be written as $w(1; i, \boldsymbol{X}_t) = \frac{1}{N_b}\sum_{n=1}^{N_b}\boldsymbol{X}_t^{[n]}(i) =: w(i, \boldsymbol{X}_t)$ and $w(\boldsymbol{X}_t) := (w(1, \boldsymbol{X}_t), \dots, w(L, \boldsymbol{X}_t))$.

For the proof, we let $w(\boldsymbol{X}_t), t = 0, 1, \dots,$ move over a sequence of increasing levels w_t^*. These levels are defined in such a way that $f(\boldsymbol{X}_t)$ is strictly decreasing and at the same time the update of π_t can be controlled such that we are able to give a lower bound on the probability for this to happen.

For $t = 0, \dots, L - 1$ let

$$w_t^* := \big(w_t^*(1), \dots, w_t^*(L)\big) := (1, \dots, 1, \alpha_t, \dots, \alpha_t) \quad \text{(V.18)}$$

with $t + 1$ entries '1' and some $\alpha_t \in (0, 1)$ defined in (V.21). We write $w(\boldsymbol{X}_t) \succeq w_t^*$ if and only if $w(i, \boldsymbol{X}_t) = 1$ for $i = 1, \dots, t{+}1$ and $w(i, \boldsymbol{X}_t) \geq \alpha_t$ for $i = t{+}2, \dots, L$. For $t = L{-}1$, $w(\boldsymbol{X}_t) \succeq w_t^*$ means that $\mathcal{N}_b$ consists of the optimal solution $s^* = (1, \dots, 1)$ only. Hence we have

$$\begin{aligned} &\mathbf{P}(\tau < L) \geq \mathbf{P}(w(\boldsymbol{X}_{L-1}) \succeq w_{L-1}^*) \qquad \text{(V.19)}\\ &\geq \mathbf{P}\big(w(\boldsymbol{X}_0) \succeq w_0^*, \dots, w(\boldsymbol{X}_{L-1}) \succeq w_{L-1}^*\big)\\ &= \mathbf{P}(w(\boldsymbol{X}_0) \succeq w_0^*) \cdot \\ &\prod_{t=1}^{L-1}\mathbf{P}\big[w(\boldsymbol{X}_t) \succeq w_t^* \mid w(\boldsymbol{X}_m) \succeq w_m^*, m = 0, \dots, t - 1\big]. \end{aligned}$$

We show below that there are constants $a, b, c > 0$ such that for all L large enough and for all N

$$\begin{aligned} &\mathbf{P}\big[w(\boldsymbol{X}_t) \succeq w_t^* \mid w(\boldsymbol{X}_m) \succeq w_m^*, m = 0, \dots, t - 1\big]\\ &\quad \geq (1 - e^{-aN})\big(1 - e^{-b\frac{N}{L^2} + c}\big)^{L-t-1}. \qquad \text{(V.20)} \end{aligned}$$

With $N = L^{2+\epsilon}$ for some $\epsilon > 0$ we may then conclude that

$$\mathbf{P}(\tau < L) \geq (1 - e^{-aL^{2+\epsilon}})^L\big(1 - e^{-bL^{\epsilon} + c}\big)^{(L^2 - L)/2}.$$

Now this last bound can be shown to converge to 1 for $L \to \infty$ using standard methods from analysis. Hence the proof of Theorem 2 is complete, once we have shown (V.20) for $t = 0, \dots, L - 1$.

In the first step we show how the levels w_t^* influence π_t. We abbreviate the event $[w(\boldsymbol{X}_m) \succeq w_m^*, m = 0, \dots, t - 1]$ by $\mathfrak{W}_t$, $\mathfrak{W}_0$ indicating the empty condition. Let

$$\alpha_t := \frac{1}{2}(1 - \frac{1}{L})^{t+1}, t = 0, \dots, L{-}1, \text{ and } \alpha_{-1} := \frac{1}{2}. \quad \text{(V.21)}$$

Conditioned on $\mathfrak{W}_t$,

$$\pi_t(i) \geq \begin{cases} 1 - (1 - \varrho)^{t-i+1} & \text{for } 1 \leq i \leq t \\ \alpha_{t-1} & \text{for } t < i \leq L \end{cases} \quad \text{(V.22)}$$

for $t = 0, \dots, L - 1$. The proof is by induction on t using the basic recursion (II.7) and the property

$$\begin{aligned} &w(i, \boldsymbol{X}_m) = 1 \quad \text{for all } m = i - 1, i, \dots, t - 1, \text{ if } i \leq t,\\ &w(i, \boldsymbol{X}_m) \geq \alpha_m \quad \text{for all } m = 0, 1, \dots, t - 1, \text{ if } i > t, \end{aligned}$$

which follows from the definition of w_t^* under $\mathfrak{W}_t$. Let $v_{t+1} = \mathbf{P}\big[\boldsymbol{X}_t^{(n)}(1) = \dots = \boldsymbol{X}_t^{(n)}(t{+}1) = 1 \mid \mathfrak{W}_t\big]$ be the probability to sample a solution that has $t{+}1$ leading 1s, then from (V.22) we obtain

$$v_{t+1} = \prod_{i=1}^{t+1}\pi_t(i) \geq \frac{1}{3e}\prod_{m=1}^{\infty}\big(1 - (1 - \varrho)^m\big) =: \kappa(\varrho) \quad \text{(V.23)}$$

for L large enough, as then $\alpha_t \geq 1/(3e)$.

Now we want to determine simple conditions on $\boldsymbol{X}_t$ that imply $w(\boldsymbol{X}_t) \succeq w_t^*$. To do so, we look at the matrix $\boldsymbol{X}_t$ *columnwise* observing the independence of its entries.

Let $M^{(t+1)}$ be the number of rows with at least $t{+}1$ leading 1s, then $w(i, \boldsymbol{X}_t) = 1, i = 1, \dots, t{+}1$, requires that $M^{(t+1)} \geq N_b$. These $M^{(t+1)}$ rows are also the first rows in $\boldsymbol{X}_t^{[\cdot]}$. Next, $w(i, \boldsymbol{X}_t) \geq \alpha_t$ requires for the number of 1s in column $i = t + 2$

$$Y^{(i)} := \sum_{n=1}^{M^{(i-1)}}\boldsymbol{X}_t^{[n]}(i) \;\geq\; \alpha_t N_b. \quad \text{(V.24)}$$

Here we have to restrict the number of rows to the present 'candidate' rows $1, \dots, M^{(i-1)}$ from which the set $\mathcal{N}_b$ is selected. After looking at column i in this way, we define

$$M^{(i)} := \max\{N_b, Y^{(i)}\} \quad \text{(V.25)}$$

and repeat (V.24), (V.25) for $i = t + 3, \dots, L$. We then obtain

$$\begin{aligned} &\mathbf{P}\big[w(\boldsymbol{X}_t) \succeq w_t^* \mid \mathfrak{W}_t\big] \qquad \text{(V.26)}\\ &\geq \mathbf{P}\big[M^{(t+1)} \geq N_b, Y^{(i)} \geq \alpha_t N_b, i = t + 2, \dots, L \mid \mathfrak{W}_t\big]\\ &= \mathbf{P}\big[M^{(t+1)} \geq N_b \mid \mathfrak{W}_t\big] \cdot \prod_{i=t+2}^{L}\mathbf{P}\big[Y^{(i)} \geq \alpha_t N_b \end{aligned}$$

$$\mid Y^{(l)} \geq \alpha_t N_b, l = t{+}2, \ldots, i{-}1, M^{(t+1)} \geq N_b, \mathfrak{W}_t \big]$$

To derive the desired lower bounds for these expressions we need the Chernoff bound (see e.g. [14], Theorem 4.2) in the following form: let $Z_1, \ldots, Z_m$ be i.i.d. 0-1-distributed with success probability p, then for any $0 < r < mp$ we have

$$\mathbf{P}\Big(\sum_{i=1}^{m} Z_i \ \leq\ r\Big) \leq e^{-\frac{1}{2}(1-\frac{r}{mp})^2 mp}. \tag{V.27}$$

Conditioned on $\mathfrak{W}_t$, $M^{(t+1)}$ is distributed as the number of successes in a row of N i.i.d. experiments, each with success probability $p := v_{t+1}$. We obtain for $\beta < \kappa(\varrho) \leq v_{t+1}$ that $N_b = \lfloor \beta N \rfloor \leq \beta N < v_{t+1} N$. Hence for $t = 0, \ldots, L-1$

$$\begin{aligned} &\mathbf{P}\big[M^{(t+1)} \geq N_b \mid \mathfrak{W}_t\big] = 1 - \mathbf{P}\big[M^{(t+1)} < N_b \mid \mathfrak{W}_t\big] \\ &\geq 1 - \mathbf{E}\big[e^{-\frac{1}{2}(1-\frac{N_b}{v_{t+1}N})^2 v_{t+1} N} \mid \mathfrak{W}_t\big] \\ &\geq 1 - e^{-\frac{1}{2}(1-\frac{\beta}{\kappa(\varrho)})^2 \kappa(\varrho) N}, \end{aligned} \tag{V.28}$$

where we used (V.27). Hence, in (V.20) we may define $a := \frac{1}{2}(1 - \frac{\beta}{\kappa(\varrho)})^2 \kappa(\varrho)$. Similarly, $Y^{(t+2)}$ is distributed as the number of 1s in $M^{(t+1)}$ i.i.d. trials each with success probability $\pi_t(t+2)$. From (V.22) and the definition of α_t we see that under the condition used in (V.26) we have $\pi_t(t+2)M^{(t+1)} \geq \alpha_{t-1} M^{(t+1)} > \alpha_t N_b$. Using the Chernoff bound we therefore obtain, for L large enough,

$$\begin{aligned} &\mathbf{P}\big[Y^{(t+2)} \geq \alpha_t N_b \mid \mathfrak{W}_t, M^{(t+1)} \geq N_b\big] \\ &= 1 - \mathbf{P}\big[Y^{(t+2)} < \alpha_t N_b \mid \mathfrak{W}_t, M^{(t+1)} \geq N_b\big] \\ &\geq 1 - e^{-\frac{1}{2}(1-\frac{\alpha_t}{\alpha_{t-1}})^2 \frac{N_b}{3e}} = 1 - e^{-\frac{N_b}{6eL^2}} \geq 1 - e^{-\frac{\beta N - 1}{6eL^2}}, \end{aligned} \tag{V.29}$$

where we used $\frac{\alpha_t}{\alpha_{t-1}} = 1 - \frac{1}{L}$. A completely analogous derivation holds for the other factors in (V.26) with $i = t{+}3, \ldots, L$. Hence, with $b := \frac{\beta}{6e}, c := \frac{1}{6e}$, we see from (V.29), (V.28) and (V.26) that (V.20) holds for L large enough. ■

D. Proof of Theorem 5

The proof of Theorem 5 about convergent samples uses an inductive argument. The induction hypothesis assumes, that the first i positions of all samples have already converged against a fixed value $y \in R_i$ and we show that then also the sampled value on the $i+1$st position will become constant in finite time. More precisely, we assume as induction hypothesis that for a fixed $i \in I$ the following holds with probability one

$$\lim_{t\to\infty} \boldsymbol{X}_t^{(n)}(1, \ldots, i) = y$$

for all $n = 1, \ldots, N$ and some $y \in R_i$. As R_i, the set of possible values of $X_m^{(n)}(1, \ldots, i)$ is finite, this condition is equivalent to requiring that there are random variables y and T taking on values in R_i and $\mathbb{N}$, defined on the common probability space, such that the following event has probability one:

$$\forall m \geq 0 \ \forall n = 1, \ldots, N \quad \boldsymbol{X}_{T+m}^{(n)}(1, \ldots, i) = y. \tag{V.30}$$

Lemmas 11 and 12 below contain the crucial induction step to prove convergence of samples. Lemma 11 shows that under condition (V.30) we may apply the recursion of Lemma 7 c) to $Q'_{\mathbf{\Pi}_t}$. We use the following definitions for $\boldsymbol{p} \in \mathbb{P}(\mathcal{A})^L$, $i \in I$ and $y \in R_i$

$$G_i(y, \boldsymbol{p}) := \sum_{a' \in C_i(y)} \boldsymbol{p}(a'; i+1) \quad \text{and} \tag{V.31}$$

$$\varrho_t^y := \frac{\varrho_t}{G_i(y, \mathbf{\Pi}_t)} \quad \text{for } t \geq 1. \tag{V.32}$$

From (V.5) we see that always $G_i(y, \mathbf{\Pi}_t) > 0$.

Lemma 11. Let $i \in I$ be fixed and assume that (V.30) holds with probability one for random variables y, T as above. Then the following holds almost surely for all $m \geq 1$

- **a)** $G_i(y, \mathbf{\Pi}_{T+m}) = 1 - \big(1 - G_i(y, \mathbf{\Pi}_T)\big) \prod_{l=1}^{m} (1 - \varrho_{T+l})$, and $G_i(y, \mathbf{\Pi}_{T+m})$ is an increasing function of m .
- **b)** $0 < \varrho_{T+m} \leq \varrho_{T+m}^y < 1$.
- **c)** For all $a \in C_i(y)$ we have
$$\begin{aligned} Q'_{\mathbf{\Pi}_{T+m}}(a; i+1, y) &= (1 - \varrho_{T+m}^y) Q'_{\mathbf{\Pi}_{T+m-1}}(a; i+1, y) \\ &+ \varrho_{T+m}^y w(a; i+1, \boldsymbol{X}_{T+m-1}). \end{aligned} \tag{V.33}$$
Thus the implications of Lemma 7 c) hold with $q_t := Q'_{\mathbf{\Pi}_{T+t}}(a; i+1, y)$, $r_t := \varrho_{T+t}^y$ and $w_t := w(a; i+1, \boldsymbol{X}_{T+t})$ with the restriction that the strict right hand bound in (V.3) only holds if $|C_i(y)| > 1$.
- **d)** If $w(a; i{+}1, \boldsymbol{X}_{T+l}) = w \in [0, 1]$ for all $l = 0, \ldots, m-1$, then
$$Q'_{\mathbf{\Pi}_{T+m}}(a; i+1, y)$$
$$= w - \big(w - Q'_{\mathbf{\Pi}_T}(a; i{+}1, y)\big) \prod_{l=1}^{m} (1 - \varrho_l^y) \tag{V.34}$$
$$\geq w - \big(w - Q'_{\mathbf{\Pi}_T}(a; i{+}1, y)\big) \prod_{l=1}^{m} (1 - \varrho_l). \tag{V.35}$$

In fact, the following proof shows that, for the assertions of Lemma 11 to hold for a fixed $m \geq 1$, instead of (V.30) the following weaker condition is sufficient

$$\forall l = 0, \ldots, m-1 \ \forall n = 1, \ldots, N \quad \boldsymbol{X}_{T+l}^{(n)}(1, \ldots, i) = y. \tag{V.36}$$

Proof of Lemma 11: **a)** From the basic recursion (II.7), we have for any $m \geq 1$

$$\begin{aligned} G_i(y, \mathbf{\Pi}_{T+m}) &= \sum_{a' \in C_i(y)} \mathbf{\Pi}_{T+m}(a'; i+1) \\ &= (1 - \varrho_{T+m}) G_i(y, \mathbf{\Pi}_{T+m-1}) + \varrho_{T+m}, \end{aligned} \tag{V.37}$$

as $\sum_{a' \in C_i(y)} w(a'; i+1, \boldsymbol{X}_{T+m-1}) = 1$. Hence, $q_t := G_i(y, \mathbf{\Pi}_{T+t})$ fulfills the condition (V.1) of Lemma 7 with $w_m \equiv 1$. Now (V.4) shows that

$$G_i(y, \mathbf{\Pi}_{T+m}) = 1 - \big(1 - G_i(y, \mathbf{\Pi}_T)\big) \prod_{l=1}^{m} (1 - \varrho_{T+l}).$$

Also, from (V.37) we have $G_i(y, \mathbf{\Pi}_{T+m+1}) \geq G_i(y, \mathbf{\Pi}_{T+m})$.

b) We have $0 < \varrho_{T+m} \leq \varrho_{T+m}^y$ and $G_i(y, \mathbf{\Pi}_{T+m-1}) > 0$ by (V.5). From (V.37) we now see $\varrho_{T+m} < G_i(y, \mathbf{\Pi}_{T+m})$, hence $\varrho_t^y < 1$.

c) From (V.37) we obtain

$$(1 - \varrho_{T+m}) G_i(y, \mathbf{\Pi}_{T+m-1}) = G_i(y, \mathbf{\Pi}_{T+m}) - \varrho_{T+m}.$$

This together with the basic recursion for $\mathbf{\Pi}_t$ shows that $Q'_{\mathbf{\Pi}_{T+m}}$ fulfills the recursion

$$\begin{aligned}Q'_{\mathbf{\Pi}_{T+m}}(a;i+1,y) &= \frac{\mathbf{\Pi}_{T+m}(a;i+1)}{G_i(y,\mathbf{\Pi}_{T+m})}\\ &= \frac{(1-\varrho_{T+m})\mathbf{\Pi}_{T+m-1}(a;i+1)}{G_i(y,\mathbf{\Pi}_{T+m})}\\ &\quad + \frac{\varrho_{T+m}\, w(a;i+1,\boldsymbol{X}_{T+m-1})}{G_i(y,\mathbf{\Pi}_{T+m})}\\ &= (1-\varrho^y_{T+m})Q'_{\mathbf{\Pi}_{T+m-1}}(a;i+1,y)\\ &\quad + \varrho^y_{T+m}\, w(a;i+1,\boldsymbol{X}_{T+m-1}).\end{aligned}$$

For the application of Lemma 7 c), we only have to check that $0 < r_t = \varrho^y_t < 1$, which was shown in part b), and $0 < q_0 = Q_{\mathbf{\Pi}_T}(a;i+1,y) < 1$ for $|C_i(y)| > 1$, which was shown in Lemma 9 b).

d) From part c) and Lemma 7 (V.4) with $q_0 = Q'_{\mathbf{\Pi}_T}(a;i+1,y)$ we obtain

$$Q'_{\mathbf{\Pi}_{T+m}}(a;i{+}1,y) = w - \big(w - Q'_{\mathbf{\Pi}_T}(a;i{+}1,y)\big)\prod_{l=1}^{m}(1-\varrho^y_l)$$

such that the assertion follows from part b). ■

Lemma 12 contains the induction step for the proof of Theorem 5.

Lemma 12. Assume that for some $\varrho > 0$ we have

$$\varrho_t \geq \varrho > 0 \quad \text{for all } t \geq 1, \tag{V.38}$$

and assume, as in Lemma 11, that there are $i \in I$ and random variables y, T such that (V.30) holds with probability one. Then the following holds almost surely: there are $T' \geq 0$ and $a_0 \in \mathcal{A}$ with

$$\boldsymbol{X}^{(n)}_{T'+m}(i+1) = a_0 \quad \text{for all } n = 1,\ldots,N \text{ and all } m \geq 0. \tag{V.39}$$

This Lemma says that, if all samples have a common leading partial solution y of length i after a finite number of iterations, then the samples will finally also coincide on the next position $i+1$, if ϱ_t is bounded away from 0. Note that the condition is always fulfilled for $i := 0$, $y := \diamond$. Thus iterated application of this Lemma allows to prove Theorem 5 below.

Proof of Lemma 12: Part of this proof extends the approach used in [3] to prove convergence of densities and closes a small gap in that proof.

As the proof is quite long, we separate it into several labeled steps. Let $i \in I$ be fixed and let y and T be such that (V.30) holds with probability one. Note that if $|C_i(y)| = 1$, the conclusion trivially holds, so we may assume $|C_i(y)| > 1$. To show the final assertion of the Lemma we have to prove:
(ConvQ) There is a (random variable) a_0 taking on values in $\mathcal{A}$ such that almost surely

$$\lim_{t\to\infty} Q_{\mathbf{\Pi}_t}(a_0;i+1,y) = \lim_{t\to\infty} Q'_{\mathbf{\Pi}_t}(a_0;i+1,y) = 1.$$

(ConvQ) will follow, after we have shown convergence of the relative frequencies:

(ConvW) There is a (random variable) a_0 as above such that almost surely

$$\lim_{t\to\infty} w(a;i+1,X_t) = \begin{cases} 1 & \text{if } a = a_0\\ 0 & \text{if } a \neq a_0\end{cases} \quad \text{for all } a \in \mathcal{A}.$$

(ConvW) in turn follows, if we have shown that $w(a;i{+}1,X_t)$ finally becomes monotone:
(MonW) For all $a \in C_i(y)$, it holds almost surely that

$$Z(a;m) := w(a;i+1,\boldsymbol{X}_{T+m}) - w(a;i+1,\boldsymbol{X}_{T+m-1})$$

has only finitely many sign changes as function of m.

We now prove these three steps in reverse order. Recall that under the condition of the Lemma, almost all solutions from $\boldsymbol{X}_{T+m}$ and $\boldsymbol{X}_{T+m-1}$ coincide in their first i positions.

To prove **(MonW)** let $M_k(a)$ be the k-th m such that $Z(a;m+1)Z(a;m) < 0$. As $\mathcal{A}$ is finite, (MonW) holds if for any fixed $a \in C_i(y)$ and $M_k := M_k(a)$

$$\mathbf{P}\big(\forall k \in \mathbb{N} \quad M_k < \infty\big) = 0. \tag{V.40}$$

Now, observe that

$$\begin{aligned}&\mathbf{P}\big(\forall k \in \mathbb{N} \quad M_k < \infty\big)\\ &\quad = \mathbf{P}(M_1 < \infty)\prod_{k=2}^{\infty}\mathbf{P}\big[M_k < \infty \mid M_{k-1} < \infty\big]\\ &\quad \leq \prod_{k=2}^{\infty}\big(1 - \mathbf{P}\big[M_k = \infty \mid M_{k-1} < \infty\big]\big).\end{aligned} \tag{V.41}$$

We are going to show that $\mathbf{P}\big[M_k = \infty \mid M_{k-1} < \infty\big] \geq \kappa > 0$ for a constant lower bound $\kappa > 0$, this proves (V.40). We have

$$\begin{aligned}&\mathbf{P}\big[M_k = \infty \mid M_{k-1} < \infty\big] =\\ &\sum_{d=k-1}^{\infty}\mathbf{P}\big[M_k = \infty \mid M_{k-1} = d\big]\cdot\mathbf{P}\big[M_{k-1} = d \mid M_{k-1} < \infty\big].\end{aligned}$$

If $M_{k-1} = d$ then $Z(a;d+1) \neq 0$, hence

$$\begin{aligned}&\mathbf{P}\big[M_k = \infty \mid M_{k-1} = d\big]\\ &\quad = \mathbf{P}\big[M_k = \infty \mid M_{k-1} = d, Z(a;d+1) > 0\big]\\ &\qquad\qquad \cdot\mathbf{P}\big[Z(a;d+1) > 0 \mid M_{k-1} = d\big]\\ &\quad + \mathbf{P}\big[M_k = \infty \mid M_{k-1} = d, Z(a;d+1) < 0\big]\\ &\qquad\qquad \cdot\mathbf{P}\big[Z(a;d+1) < 0 \mid M_{k-1} = d\big].\end{aligned} \tag{V.42}$$

We use the abbreviation $W_m := w(a;i+1,\boldsymbol{X}_{T+m})$, $\mathfrak{M}$ for the event $[M_{k-1} = d, Z(a;d+1) > 0]$, and $\mathfrak{W}_m$ for $[W_l = 1, l = d+2,\ldots,m-1]$. Conditioned on $\mathfrak{M}$, the event $M_k = \infty$ is implied by $W_m = 1, m \geq d{+}2$. We can therefore give the following rough bound for the first term in (V.42)

$$\begin{aligned}&\mathbf{P}\big[M_k = \infty \mid M_{k-1} = d, Z(a;d+1) > 0\big]\\ &\geq \mathbf{P}\Big[W_m = 1, m \geq d+2 \mid \mathfrak{M}\Big]\\ &= \mathbf{P}\Big[W_{d+2} = 1 \mid \mathfrak{M}\Big]\prod_{m=d+3}^{\infty}\mathbf{P}\Big[W_m = 1 \mid \mathfrak{W}_m, \mathfrak{M}\Big]\\ &\geq \mathbf{E}\Big[\big[Q_{\mathbf{\Pi}_{T+d+2}}(a;i+1,y)\big]^N \mid \mathfrak{M}\Big]\end{aligned} \tag{V.43}$$

$$\cdot \prod_{m=d+3}^{\infty} \mathbf{E}\Big[\big[Q_{\mathbf{\Pi}_{T+m}}(a;i+1,y)\big]^N \;\Big|\; \mathfrak{W}_m, \mathfrak{M}\Big].$$

From Lemma 9 a) and (V.44) below, we get for the first factor in (V.43)

$$\mathbf{E}\Big[\big[Q_{\mathbf{\Pi}_{T+d+2}}(a;i+1,y)\big]^N \mid \mathfrak{M}\Big] \geq \mathbf{E}\Big[\big[\ell\big(Q'_{\mathbf{\Pi}_{T+d+2}}(a;i+1,y)\big)\big]^N \mid \mathfrak{M}\Big] \geq \ \ell\big(\frac{\varrho}{N_b}\big)^N,$$

where we used the fact that $\mathfrak{M}$ implies $Z(a;d+1) > 0$. Therefore $w(a;i+1,\boldsymbol{X}_{T+d+1}) > w(a;i+1,\boldsymbol{X}_{T+d}) \geq 0$ and $w(a;i+1,\boldsymbol{X}_{T+d+1}) \geq \frac{1}{N_b}$. Lemma 11 c) now shows

$$Q'_{\mathbf{\Pi}_{T+d+2}}(a;i+1,y) \geq \frac{\varrho^y_{T+d+2}}{N_b} \geq \frac{\varrho}{N_b} > 0. \qquad \text{(V.44)}$$

For the second factor in (V.43), we see from Lemma 11 d) that, under the condition $W_l = 1, l = d+2, \ldots, m-1$, we have

$$Q'_{\mathbf{\Pi}_{T+m}}(a;i+1,y) \geq 1 - (1-\varrho)^{m-d-2}.$$

Hence, in the second term of (V.43) the integrand can be bounded for all $m \geq d+3$ in the following way

$$\big[Q_{\mathbf{\Pi}_{T+m}}(a;i+1,y)\big]^N \geq \big[\ell\big(Q'_{\mathbf{\Pi}_{T+m}}(a;i+1,y)\big)\big]^N \geq \big[\ell\big(1-(1-\varrho)^{m-d-2}\big)\big]^N = \Big[1 - h\big((1-\varrho)^{m-d-2}\big)\Big]^N.$$

Collecting the equations above, we now obtain for all $k, d \in \mathbb{N}$

$$\begin{aligned}\mathbf{P}\big[M_k = \infty \mid M_{k-1} = d, Z(a;d+1) > 0\big] \qquad \text{(V.45)}\\ \geq \ell\big(\frac{\varrho}{N_b}\big)^N \prod_{m=d+3}^{\infty} \Big[1 - h\big((1-\varrho)^{m-d-2}\big)\Big]^N \\ = \ell\big(\frac{\varrho}{N_b}\big)^N \prod_{m=1}^{\infty} \Big[1 - h\big((1-\varrho)^{m}\big)\Big]^N =: \kappa.\end{aligned}$$

From Lemma 8 d), we see that $\sum_{m=1}^{\infty}(1-\varrho)^m < \infty$ implies $\sum_{m=1}^{\infty} h\big((1-\varrho)^m\big) < \infty$, hence $\kappa > 0$.

In a completely analogous manner, using the events $W_m = 0, m \geq d+2$, we can also show that

$$\mathbf{P}\big[M_k = \infty \mid M_{k-1} = d, Z(a;d+1) < 0\big] \geq \kappa,$$

which now proves (V.40) and (MonW).

We are now going to prove **(ConvW)**. From (MonW) we know that $m \mapsto w(a;i+1,\boldsymbol{X}_{T+m})$ must eventually be monotonic and therefore converge to one of the values in $\mathcal{W} := \{0, \frac{1}{N_b}, \ldots, \frac{N_b-1}{N_b}, 1\}$ (remember that $w(\cdot)$ is a relative frequency from a sample of size N_b). Hence, there is a random variable $T' \geq T$, and for each $a \in \mathcal{A}$ a random variable V_a taking on values in $\mathcal{W}$, such that with probability 1, $w(a;i+1,\boldsymbol{X}_{T'+m}) = V_a$ for all $m \geq 0$.

>From the recursion (V.33) and Lemma 7 c) and b), we see that, under our conditions for all $m \geq 0$ and $a \in C_i(y)$,

$$Q'_{\mathbf{\Pi}_{T'+m}}(a;i+1,y) = Q'_{\mathbf{\Pi}_{T'}}(a;i+1,y) \cdot \prod_{l=1}^{m}(1-\varrho^y_{T'+l}) + V_a\big(1 - \prod_{l=1}^{m}(1-\varrho^y_{T'+l})\big). \qquad \text{(V.46)}$$

From (V.38) and $\varrho^y_{T'+l} \geq \varrho_{T'+l} > \varrho$, we have $\sum_{l=1}^{\infty} \varrho^y_{T'+l} = \infty$ and hence $\prod_{l=1}^{\infty}(1-\varrho^y_{T'+l}) = 0$. Therefore, (V.46) implies that, almost surely,

$$\lim_{t\to\infty} Q'_{\mathbf{\Pi}_t}(a;i+1,y) = V_a \ \text{ for all } a \in C_i(y). \qquad \text{(V.47)}$$

The bounds from Lemma 9 a) and the continuity of the bounding functions h and ℓ (see Lemma 8 a)) lead us to conclude that for all $a \in C_i(y)$

$$\begin{aligned}\ell(V_a) &\leq \liminf_{t\to\infty} Q_{\mathbf{\Pi}_t}(a;i+1,y) \qquad \text{(V.48)}\\ &\leq \limsup_{t\to\infty} Q_{\mathbf{\Pi}_t}(a;i+1,y) \leq h(V_a).\end{aligned}$$

We want to show next, that the limit V_a can take on values in $\{0,1\}$ only, more precisely:

$$\exists a_0 \in C_i(y)\ V_{a_0} = 1 \text{ and } \forall a \in \big(C_i(y) - \{a_0\}\big)\ V_a = 0 \qquad \text{(V.49)}$$

holds almost surely. To prove (V.49), we first show that for all $a \in C_i(y)$ we have $\mathbf{P}(V_a \in (0,1)) = 0$. We again use the abbreviation $W_m = w(a;i+1,\boldsymbol{X}_{T+m})$ and $\mathcal{W}_0 := \mathcal{W} \cap (0,1) = \{\frac{1}{N_b}, \frac{2}{N_b}, \ldots, 1 - \frac{1}{N_b}\}$. We then have

$$\begin{aligned}\mathbf{P}(V_a \in (0,1)) &= \mathbf{P}(\lim_{k\to\infty} W_k \in \mathcal{W}_0) \qquad \text{(V.50)}\\ &= \lim_{k\to\infty}\Big(\mathbf{P}\big(W_k \in \mathcal{W}_0\big) \cdot \prod_{m=k+1}^{\infty} \mathbf{P}\big[W_m \in \mathcal{W}_0 \mid W_l \in \mathcal{W}_0, l = k, \ldots, m-1\big]\Big)\\ &\leq \lim_{k\to\infty} \prod_{m=k+1}^{\infty} \mathbf{P}\big[W_m \in \mathcal{W}_0 \mid W_l \in \mathcal{W}_0, l = k, \ldots, m-1\big].\end{aligned}$$

Writing $\mathfrak{W}$ for $W_l \in \mathcal{W}_0, l = k, \ldots, m-1$, we have for $m > k$

$$\begin{aligned}&\mathbf{P}[W_m \in \mathcal{W}_0 \mid W_l \in \mathcal{W}_0, l = k, \ldots, m-1] \qquad \text{(V.51)}\\ &= 1 - \mathbf{E}\big[\big(1 - Q_{\mathbf{\Pi}_{T+m}}(a;i+1,y)\big)^N \mid \mathfrak{W}\big]\\ &\leq 1 - \Big(1 - h\big((1-\varrho)^{m-k} + 1 - \frac{1}{N_b}\big)\Big)^N,\end{aligned}$$

where we used the fact that under $\mathfrak{W}$ we have $W_l \leq 1 - \frac{1}{N_b}, l = k, \ldots, m-1$. Hence we can see as in (V.46), that for $m > k$

$$\begin{aligned}&Q'_{\mathbf{\Pi}_{T+m}}(a;i+1,y) \leq Q'_{\mathbf{\Pi}_{T+k}}(a;i+1,y) \cdot \qquad \text{(V.52)}\\ &\prod_{l=k+1}^{m}(1-\varrho^y_{T+l}) + (1 - \frac{1}{N_b})\big(1 - \prod_{l=k+1}^{m}(1-\varrho^y_{T+l})\big)\\ &\leq \prod_{l=k+1}^{m}(1-\varrho^y_{T+l}) + 1 - \frac{1}{N_b} \leq (1-\varrho)^{m-k} + 1 - \frac{1}{N_b}\end{aligned}$$

But then, for m large enough, the upper bound of the probability (V.51) is smaller than 1, and the infinite product in (V.50) vanishes. Therefore, we have shown

$$\mathbf{P}\big(V_a \in \{0,1\} \ \text{ for all } a \in C_i(y)\big) = 1. \qquad \text{(V.53)}$$

We also know that, $\mathbf{P}$-almost surely,

$$1 = \sum_{a \in C_i(y)} W_m = \sum_{a \in C_i(y)} w(a;i+1,\boldsymbol{X}_{T+m})$$

for all $m \geq 0$. Hence, this must also hold for the limits: $\mathbf{P}\big(\sum_{a \in C_i(y)} V_a = 1\big) = 1$. This together with (V.53) proves (V.49) and hence (ConvW).

Now **(ConvQ)** immediately follows: from (V.47) we see that there is a random variable a_0 such that

$$\mathbf{P}\big(\lim_{t\to\infty} Q'_{\mathbf{\Pi}_t}(a_0; i+1, y) = 1\big) = 1. \tag{V.54}$$

From $h(1) = \ell(1) = 1$ (see Lemma 8 b)) and (V.48) we see that the same must hold for $Q_{\mathbf{\Pi}_t}(a_0; i+1, y)$.

Thus we have proved (ConvQ) and we are now going to show that this (more precisely (V.54)) implies the assertion of the Lemma, i.e. $\boldsymbol{X}_m^{(n)}(i+1) = a_0, n = 1, \dots, N$, for large enough m, $\mathbf{P}$-almost surely. In other words, the next position $i+1$ also becomes identical in all samples after finitely many iterations.

We write $[\boldsymbol{X}_t^{(\cdot)}(i{+}1) \equiv a_0]$ for the event $[\boldsymbol{X}_t^{(n)}(i{+}1) = a_0, n = 1, \dots, N]$ that all samples have the same $i+1$-st component. We then have

$$\begin{aligned} &\mathbf{P}\Big(\exists k \in \mathbb{N}\ \forall m \geq k \quad X_m^{(\cdot)}(i{+}1) \equiv a_0\Big) \qquad \text{(V.55)}\\ &= \lim_{k\to\infty} \Big(\mathbf{P}\big(X_k^{(\cdot)}(i{+}1) \equiv a_0\big) \cdot \prod_{m=k+1}^{\infty} \\ &\quad \mathbf{P}\big[X_m^{(\cdot)}(i{+}1) \equiv a_0 \big| X_l^{(\cdot)}(i{+}1) \equiv a_0, l = k, \dots, m-1\big]\Big). \end{aligned}$$

Now from (V.54), using bounded convergence, we see that for the first factor

$$\begin{aligned} \lim_{k\to\infty} \mathbf{P}\big(X_k^{(\cdot)}(i{+}1) \equiv a_0\big) &= \lim_{k\to\infty} \mathbf{E}\big(Q_{\mathbf{\Pi}_k}(a_0; i{+}1, y)^N\big) \\ &= \mathbf{E}\Big(\lim_{k\to\infty} \big(Q_{\mathbf{\Pi}_k}(a_0; i{+}1, y)\big)^N\Big) = 1. \qquad \text{(V.56)} \end{aligned}$$

For the second expression in (V.55), we obtain for $m \geq k \geq T$

$$\begin{aligned} &\mathbf{P}\big[X_m^{(\cdot)}(i{+}1) \equiv a_0 \;\big|\; X_l^{(\cdot)}(i{+}1) \equiv a_0, l = k, \dots, m-1\big] \\ &= \mathbf{E}\Big[\big(Q_{\mathbf{\Pi}_m}(a_0; i{+}1, y)\big)^N \;\Big| \\ &\qquad X_l^{(\cdot)}(i{+}1) \equiv a_0, l = k, \dots, m-1\Big]. \qquad \text{(V.57)} \end{aligned}$$

We want to bound this last expression from below using $Q'_{\mathbf{\Pi}_m}$. The condition $\boldsymbol{X}_l^{(\cdot)}(i{+}1) \equiv a_0, l = k, \dots, m-1$, implies $w(a_0, i{+}1, \boldsymbol{X}_l) = 1$ for $l = k, \dots, m-1$. Therefore, we have from Lemma 11 d)

$$\begin{aligned} &Q'_{\mathbf{\Pi}_m}(a_0; i{+}1, y) \\ &\quad \geq 1 - (1 - Q'_{\mathbf{\Pi}_k}(a_0; i{+}1, y))(1 - \varrho)^{m-k}. \end{aligned}$$

Using the bounds from Lemma 9 a) and Lemma 8 for $m \geq k \geq T$, we arrive at

$$\begin{aligned} &\big[Q_{\mathbf{\Pi}_m}(a_0; i{+}1, y)\big]^N \geq \big[\ell\big(Q'_{\mathbf{\Pi}_m}(a_0; i{+}1, y)\big)\big]^N \\ &\geq \Big[1 - h\Big(\big(1 - Q'_{\mathbf{\Pi}_k}(a_0; i{+}1, y)\big)(1 - \varrho)^{m-k}\Big)\Big]^N \\ &=: \big[1 - H_{(k,m-k)}\big]^N \qquad \text{(V.58)} \end{aligned}$$

where we introduced the abbreviation $H_{(k,m-k)}$. Note that $H_{(k,m-k)}$ is independent of the condition $X_l^{(\cdot)}(i{+}1) \equiv a_0, l = k, \dots, m-1$. Combining (V.57) and (V.58) we may bound the second term in (V.55) from below by

$$\begin{aligned} &\lim_{k\to\infty} \prod_{m=k+1}^{\infty} \mathbf{E}\big((1 - H_{(k,m-k)})^N\big) \qquad \text{(V.59)} \\ &\geq \Big[\lim_{k\to\infty} \prod_{m=1}^{\infty} \Big(1 - \mathbf{E}H_{(k,m)}\Big)\Big]^N \\ &= \Big[\prod_{m=1}^{\infty} \Big(1 - \mathbf{E}\big(\lim_{k\to\infty} H_{(k,m)}\big)\Big)\Big]^N = 1, \end{aligned}$$

as $\lim_{k\to\infty} H_{(k,m)} = 0$ almost surely by (V.54). To see that lim and product may be interchanged, we use logarithms and then bounded convergence with $0 \leq -\ln(1 - \mathbf{E}H_{k,m}) \leq -\ln(1 - h((1-\varrho)^m)$ for all k.

From (V.59), (V.56) and (V.55), we now see that there is a random variable a_0 such that with probability one

$$\exists k \in \mathbb{N}\ \forall m \geq k \quad X_m^{(\cdot)}(i{+}1) \equiv a_0,$$

but this is the assertion of the Lemma. ∎

We can now prove Theorem 5 about the convergence of the samples.

Proof of Theorem 5: **a)** For $i \in I$, let Γ_i denote the event

$$\exists y \in R_i\ \forall n = 1, \dots, N \quad \lim_{t\to\infty} \boldsymbol{X}_t^{(n)}(1, \dots, i) = y,$$

and let γ_i be the event

$$\exists a_0 \in A\ \forall n = 1, \dots, N\ \lim_{t\to\infty} \boldsymbol{X}_t(i)^{(n)} = a_0.$$

In Lemma 12, we have shown that, if $\mathbf{P}(\Gamma_i) = 1$, then also $\mathbf{P}(\gamma_{i+1}) = 1$. But, as $\Gamma_i \cup \gamma_{i+1} \subset \Gamma_{i+1}$, we see that $\mathbf{P}(\Gamma_i) = 1$ implies $\mathbf{P}(\Gamma_{i+1}) = 1$. As all solutions have the empty substring $\diamond$ as partial solution, we have $\mathbf{P}(\Gamma_0) = 1$, as was noted before. But then we see from repeated application of Lemma 12 that $\mathbf{P}(\Gamma_L) = 1$, i.e. convergence of the complete samples holds with probability one.

b) We first show that convergence of samples implies

$$\sum_{m=1}^{\infty} \prod_{l=1}^{m} (1 - \varrho_l^{\diamond}) < \infty. \tag{V.60}$$

Assume that convergence of samples holds, then there must be at least one $s = (s_1, \dots, s_L) \in S$ with

$$\begin{aligned} 0 &< \mathbf{P}\big(\lim_{t\to\infty} \boldsymbol{X}_t^{(n)} = s \text{ for } n = 1, \dots, N\big) \\ &\leq \mathbf{P}\big(\lim_{t\to\infty} \boldsymbol{X}_t^{(n)}(1) = s_1 \text{ for } n = 1, \dots, N\big) \\ &= \lim_{k\to\infty} \mathbf{P}\big(\boldsymbol{X}_k^{(\cdot)}(1) \equiv s_1\big) \cdot \prod_{m=k+1}^{\infty} \\ &\qquad \mathbf{P}\big[\boldsymbol{X}_m^{(\cdot)}(1) \equiv s_1 \mid \boldsymbol{X}_l^{(\cdot)}(1) \equiv s_1, l = k, \dots, m-1\big] \\ &\leq \lim_{k\to\infty} \prod_{m=k+1}^{\infty} \mathbf{E}\big[\big(h\big(Q'_{\mathbf{\Pi}_m}(s_1; 1, \diamond)\big)\big)^N \mid \\ &\qquad\qquad w(s_1; 1, \boldsymbol{X}_l) = 1, l = k, \dots, m-1\big] \end{aligned}$$

where we used the bound of Lemma 9 a). From Lemma 11 c) and (V.3) of Lemma 7, we see that

$$\mathbf{E}\big[\big(h\big(Q'_{\mathbf{\Pi}_m}(s_1; 1, \diamond)\big)\big)^N \big| w(s_1; 1, \boldsymbol{X}_l) = 1, l = k, \dots, m-1\big]$$

$$\leq \Big(h\big(1-(1-Q'_{\mathbf{\Pi}_0}(s_1;1,\diamond))\prod_{l=1}^{m}(1-\varrho_l^{\diamond})\big)\Big)^N$$
$$= \Big(1-\ell\big((1-Q'_{\mathbf{\Pi}_0}(s_1;1,\diamond))\prod_{l=1}^{m}(1-\varrho_l^{\diamond})\big)\Big)^N.$$

Hence there must be $k \geq 0$ such that

$$0 < \prod_{m=k+1}^{\infty}\Big(1-\ell\big((1-Q'_{\mathbf{\Pi}_0}(s_1;1,\diamond))\prod_{l=1}^{m}(1-\varrho_l^{\diamond})\big)\Big).$$

By Lemma 7 a), this is only possible if

$$\sum_{m=1}^{\infty}\ell\big((1-Q'_{\mathbf{\Pi}_0}(s_1;1,\diamond))\prod_{l=1}^{m}(1-\varrho_l^{\diamond})\big) < \infty, \qquad \text{(V.61)}$$

and by Lemma 8 d), this in turn implies (V.60). Now we show that (V.60) implies that there is a positive probability to stay in non-optimal solutions forever, hence $\mathbf{P}(\tau < \infty) < 1$. From (II.9), we know that there are solutions $s = (s_1, \ldots, s_L), s' = (s'_1, \ldots, s'_L)$ with $s_1 \neq s'_1$. As we have $|S^*| = 1$, we may assume $s \in S - S^*$. Then there is a positive probability that all samples have s_1 at their first position and therefore cannot be optimal. This can be shown in a derivation similar to (V.16) to (V.17) restricted to s_1.

c) If we have convergent samples, then for some $s \in S$

$$w(a; i, \boldsymbol{X}_{T+m}) = \begin{cases} 1 & \text{if } a = s_i \\ 0 & \text{else} \end{cases}$$

for all $i = 1, \ldots, L, m \geq 0$, and hence by (V.4)

$$\mathbf{\Pi}_{T+m}(s_i; i) = 1 - \big(1 - \mathbf{\Pi}_T(s_i; i)\big)\prod_{l=1}^{m}(1-\varrho_l). \qquad \text{(V.62)}$$

Now the assertion $\lim_{m\to\infty}\mathbf{\Pi}_{T+m}(s_i;i) = 1$ follows if $\prod_{l=1}^{\infty}(1-\varrho_l) = 0$. From (V.60), we see that $\sum_{l=1}^{\infty}\varrho_l^{\diamond} = \infty$, and we may conclude from (II.8) and Lemma 11 a)

$$\lim_{l\to\infty}\frac{\varrho_l}{\varrho_l^{\diamond}} = \lim_{l\to\infty} G_0(\diamond, \mathbf{\Pi}_l) > 0.$$

Hence, we also have $\sum_{l=1}^{\infty}\varrho_l = \infty$ implying the assertion.

d) (see also [3]) From Lemma 7 (V.3) we have

$$\mathbf{\Pi}_t(a; i) \geq \mathbf{\Pi}_0(a; i)\prod_{m=1}^{t}(1-\varrho_m).$$

As $\lim_{t\to\infty}\mathbf{\Pi}_t(a;i) = 0$ must occur with positive probability for some a and i, and as $\mathbf{\Pi}_0(a;i) > 0$ by (II.8), the assertion follows. ■

E. Proof of Theorem 6

We start with a Proposition on the convergence of the density.

Lemma 13. Let $(e_k)_{k\geq 1}$ be an increasing unbounded sequence in $\mathbb{N}$ with $e_1 = 1$ and define

$$1 - \beta_k := \prod_{m=e_k+1}^{e_{k+1}-1}(1-\varrho_m), \qquad \text{for } k \geq 1.$$

If $(\varrho_t)_{t\geq 1}$ is chosen such that for some $\varrho > 0$ and all $k \geq 1$

$$\varrho_{e_k} \geq \varrho \quad \text{and} \quad \sum_{k=1}^{\infty}\beta_k < \infty, \qquad \text{(V.63)}$$

then, in the unrestricted case, the algorithm has a convergent density.

Here, the e_k represent embedded points of time at which ϱ_t is bounded away from 0, and β_k summarizes the development of ϱ_t between these points.

Proof: Fix $i \in \{1, \ldots, L\}$ and $a \in \mathcal{A}$. We have to show that

$$\lim_{t\to\infty}\mathbf{\Pi}_t(a;i) \in \{0,1\} \qquad \text{(V.64)}$$

almost surely. Note that, as there are no constraints, we have $Q_{\mathbf{\Pi}_t}(a;i,y) = \mathbf{\Pi}_t(a;i)$ for any $y \in R_{i-1}$.

The main idea of this proof is to separate the embedded time points $e_k, k \geq 1$, from the times $B_k := \{e_k+1, \ldots, e_{k+1}-1\}$ between these points. In a first step, we show that for any k and any $t \in B_k$

$$\mathbf{\Pi}_{e_k}(a;i) - \beta_k \leq \mathbf{\Pi}_t(a;i) \leq \mathbf{\Pi}_{e_k}(a;i) + \beta_k. \qquad \text{(V.65)}$$

As we assumed $\sum_{k=1}^{\infty}\beta_k < \infty$, we have $\lim_{k\to\infty}\beta_k = 0$. Convergence of the density as in (V.64) now follows, if we can prove convergence at the embedded time points, i.e.

$$\lim_{k\to\infty}\mathbf{\Pi}_{e_k}(a;i) \in \{0,1\}. \qquad \text{(V.66)}$$

This is shown in a second step.

To prove (V.65), we use the left-hand inequality of (V.3) with $q_0 := \mathbf{\Pi}_{e_k}(a;i)$ to obtain for any $t \in B_k$

$$\begin{aligned}\mathbf{\Pi}_t(a;i) &\geq \mathbf{\Pi}_{e_k}(a;i)\prod_{m=e_k+1}^{t}(1-\varrho_m)\\ &\geq \mathbf{\Pi}_{e_k}(a;i)(1-\beta_k)\\ &\geq \mathbf{\Pi}_{e_k}(a;i) - \beta_k.\end{aligned}$$

In a similar way, the second inequality in (V.65) follows from the right-hand side of (V.3).

To prove (V.66), we proceed as in the proof of Lemma 12 but restricted to the embedded times e_k: we first show that with probability one, $w(a;i,\boldsymbol{X}_{e_k-1})$ will eventually become monotonic (step (MonW) in Lemma 12) and then deduce convergence of the density $\mathbf{\Pi}_{e_k}(a;i)$ (step (ConvQ).

We abbreviate $\hat{W}_k := w(a;i,\boldsymbol{X}_{e_k-1})$, and define $\hat{Z}(k) = \hat{W}_k - \hat{W}_{k-1} = w(a;i,\boldsymbol{X}_{e_k-1}) - w(a;i,\boldsymbol{X}_{e_{k-1}-1})$. Let $\hat{M}_u$, with $u \in \mathbb{N}$, be the u-th k such that

$$\hat{Z}(k+1)\hat{Z}(k) < 0.$$

Note that $\hat{M}_u \geq u$. We show that $\mathbf{P}(\exists u \in \mathbb{N} \quad \hat{M}_u = \infty) = 1$ using the approach of (V.41) - (V.42). Therefore we have to show, as in (V.45), that there is a lower bound $\kappa > 0$ such that for all $d \geq u$ and u large enough

$$\mathbf{P}\big[\hat{M}_{u+1} = \infty \mid \hat{M}_u = d, \hat{Z}(d+1) < 0\big] \geq \kappa > 0, \qquad \text{(V.67)}$$

and similarly for $\hat{Z}(d+1) > 0$.

To prove (V.67) as in Lemma 12, we need an analogue to the basic recursion (II.7) for $\mathbf{\Pi}_{e_k}(a;i)$, which we are going to derive now. Let $\hat{\varrho}_k := \varrho_{e_k}$, then we can show that

$$\begin{aligned}(1-\hat{\varrho}_{k+1})\mathbf{\Pi}_{e_k}(a;i) + \hat{\varrho}_{k+1}\hat{W}_{k+1} - (1-\hat{\varrho}_{k+1})\beta_k \quad &\text{(V.68)}\\ \leq \mathbf{\Pi}_{e_{k+1}}(a;i) \\ \leq (1-\hat{\varrho}_{k+1})\mathbf{\Pi}_{e_k}(a;i) + \hat{\varrho}_{k+1}\hat{W}_{k+1} + (1-\hat{\varrho}_{k+1})\beta_k.\end{aligned}$$

To prove (V.68), we use the basic recursion (II.7) and the right-hand side of (V.65) to obtain

$$\begin{aligned}\mathbf{\Pi}_{e_{k+1}}(a;i) &= (1-\hat{\varrho}_{k+1})\mathbf{\Pi}_{e_{k+1}-1}(a;i) + \hat{\varrho}_{k+1}\hat{W}_{k+1}\\ &\leq (1-\hat{\varrho}_{k+1})\big[\mathbf{\Pi}_{e_k}(a;i) + \beta_k\big] + \hat{\varrho}_{k+1}\hat{W}_{k+1}\\ &= (1-\hat{\varrho}_{k+1})\mathbf{\Pi}_{e_k}(a;i) + \hat{\varrho}_{k+1}\hat{W}_{k+1} + (1-\hat{\varrho}_{k+1})\beta_k.\end{aligned}$$

In a similar way, from the left-hand inequality of (V.65), we can prove the left-hand inequality of (V.68).

An iteration of (V.68) allows to give bounds on $\mathbf{\Pi}_{e_{k+m}}$ in terms of $\mathbf{\Pi}_{e_k}$ for the case of constant $\hat{W}_{k+l} = \hat{w} \in [0,1]$ for $l = 1,\dots,m$. More precisely, we obtain from (V.68) and $\hat{\varrho}_k \geq \varrho > 0$

$$\begin{aligned}&\hat{w} + (\mathbf{\Pi}_{e_k}(a;i) - \hat{w})\prod_{l=1}^{m}(1-\hat{\varrho}_{k+l}) - \sum_{j=1}^{m}\beta_{k+m-j}(1-\varrho)^j\\ &\leq \mathbf{\Pi}_{e_{k+m}}(a;i) \leq \qquad \text{(V.69)}\\ &\hat{w} + (\mathbf{\Pi}_{e_k}(a;i) - \hat{w})\prod_{l=1}^{m}(1-\hat{\varrho}_{k+l}) + \sum_{j=1}^{m}\beta_{k+m-j}(1-\varrho)^j.\end{aligned}$$

The lengthy but simple proof of (V.69) is omitted. Combining (V.65) and (V.69) we can now bound $\mathbf{\Pi}_t(a;i)$ in terms of $\mathbf{\Pi}_{e_k}$ for all $t \in B_{k+m}$ in the case that $\hat{W}_{k+l} \equiv \hat{w}$ for all $l = 1,\dots,m$:

$$\begin{aligned}&\hat{w} + (\mathbf{\Pi}_{e_k}(a;i) - \hat{w})\prod_{l=1}^{m}(1-\hat{\varrho}_{k+l}) - \sum_{j=0}^{m}\beta_{k+m-j}(1-\varrho)^j\\ &\leq \mathbf{\Pi}_t(a;i) \leq \qquad \text{(V.70)}\\ &\hat{w} + (\mathbf{\Pi}_{e_k}(a;i) - \hat{w})\prod_{l=1}^{m}(1-\hat{\varrho}_{k+l}) + \sum_{j=0}^{m}\beta_{k+m-j}(1-\varrho)^j.\end{aligned}$$

We now return to the problem of bounding the probability in (V.67) from below. Note, that under the condition $\hat{M}_u = d$ and $\hat{Z}(d+1) < 0$, the event $\hat{W}_{d+1+l} \equiv 0$ for all $l \geq 1$ implies $\hat{M}_{u+1} = \infty$. In this case we obtain from (V.70) with $\hat{w} := 0, k := d+1$, for any $t \in B_{d+1+m}$

$$\begin{aligned}&\mathbf{\Pi}_t(a;i) \qquad \text{(V.71)}\\ &\leq \mathbf{\Pi}_{e_{d+1}}(a;i)\prod_{l=1}^{m}(1-\hat{\varrho}_{d+1+l}) + \sum_{j=0}^{m}\beta_{d+1+m-j}(1-\varrho)^j\\ &\leq (1-\varrho)^m + \sum_{j=0}^{m}\beta_{d+1+m-j}(1-\varrho)^j.\end{aligned}$$

Also, for $\hat{Z}(d+1) < 0$ we have $\hat{W}_{d+1} \leq 1 - \frac{1}{N_b}$ and, using the basic recursion (II.7), we get

$$\mathbf{\Pi}_{e_{d+1}}(a;i) \leq 1 - \frac{\hat{\varrho}_{d+1}}{N_b} \leq 1 - \frac{\varrho}{N_b},$$

and hence, by (V.65), for all $t \in B_{d+1}$

$$\mathbf{\Pi}_t(a;i) \leq 1 - \frac{\varrho}{N_b} + \beta_{d+1}. \qquad \text{(V.72)}$$

Since $\sum_{k=1}^{\infty}\beta_k < \infty$, whenever u is large enough (remember that $d = \hat{M}_u \geq u$), the upper bounds in (V.71) and (V.72) are smaller than 1.

We are now in a position to prove (V.67). We use the notation $\boldsymbol{X}_t^{(\cdot)}(i) \neq a$ as abbreviation for the event $[\boldsymbol{X}_t^{(n)}(i) \neq a$ for all $n = 1,\dots,N]$, $\hat{\mathfrak{M}}$ for $[\hat{M}_u = d, \hat{Z}(d+1) < 0]$ and $\mathfrak{W}_m$ for $[\hat{W}_l = 0, l = d+2,\dots,m-1]$. Under the condition $\hat{\mathfrak{M}}$, the event $\hat{W}_m = 0, m \geq d+2$ implies $\hat{M}_{u+1} = \infty$. Hence we have for a, i fixed as at the beginning of the proof

$$\begin{aligned}&\mathbf{P}\big[\hat{M}_{u+1} = \infty \mid \hat{M}_u = d, \hat{Z}(d+1) < 0\big] \qquad \text{(V.73)}\\ &\geq \mathbf{P}\big[\hat{W}_m = 0, m \geq d+2 \mid \hat{\mathfrak{M}}\big]\\ &= \mathbf{P}\big[\hat{W}_{d+2} = 0 \mid \hat{\mathfrak{M}}\,\big]\prod_{m=d+3}^{\infty}\mathbf{P}\big[\hat{W}_m = 0 \mid \mathfrak{W}_m, \hat{\mathfrak{M}}\,\big]\\ &\geq \mathbf{E}\Big[\big[1-\mathbf{\Pi}_{e_{d+2}-1}(a;i)\big]^N \mid \hat{\mathfrak{M}}\,\Big]\\ &\qquad \cdot \prod_{m=d+3}^{\infty}\mathbf{E}\Big[\big[1-\mathbf{\Pi}_{e_m-1}(a;i)\big]^N \,\Big|\, \mathfrak{W}_m, \hat{\mathfrak{M}}\,\Big].\end{aligned}$$

For the first factor in (V.73), we get from (V.72)

$$\begin{aligned}&\mathbf{E}\Big[\big[1-\mathbf{\Pi}_{e_{d+2}-1}(a;i)\big]^N \mid \hat{M}_u = d, \hat{Z}(d+1) < 0\Big] \quad \text{(V.74)}\\ &\geq \big[1-\big(1-\frac{\varrho}{N_b}+\beta_{d+1}\big)\big]^N = \big[\frac{\varrho}{N_b} - \beta_{d+1}\big]^N,\end{aligned}$$

which is positive for u large enough and $d \geq u$, as $\lim_{k\to\infty}\beta_k = 0$. For the second factor in (V.73), we see from (V.71)

$$\begin{aligned}&\prod_{m=d+3}^{\infty}\mathbf{E}\Big[\big[1-\mathbf{\Pi}_{e_m-1}(a;i)\big]^N \,\Big|\, \mathfrak{W}_m, \hat{\mathfrak{M}}\,\Big] \qquad \text{(V.75)}\\ &\geq \prod_{m=1}^{\infty}\Big[1-(1-\varrho)^m - \sum_{j=0}^{m}\beta_{d+1+m-j}(1-\varrho)^j\Big]^N\\ &=: \prod_{m=1}^{\infty}\Big[1-x_m(d)\Big]^N.\end{aligned}$$

Note that the time points $e_{d+2+m}-1$ belong to B_{d+1+m}, hence we may apply (V.71). We want to show that the product in (V.75) has a positive lower bound that is independent of d for u large enough (remember $d \geq u$). We have

$$\prod_{m=1}^{\infty}\big(1-x_m(d)\big) = \exp\Big(-\sum_{m=1}^{\infty} -\ln\big(1-x_m(d)\big)\Big),$$

hence it is sufficient to show that there is $\Delta < \infty$ such that for all u large enough and $d \geq u$

$$\sum_{m=1}^{\infty} -\ln\big(1-x_m(d)\big) \leq \Delta < \infty. \qquad \text{(V.76)}$$

We skip the purely technical proof here.

Together with the bound found in (V.74), we see that there is common lower bound $\kappa > 0$ for (V.73) and all $d \geq u$ when u is large enough. Therefore (V.67) holds for u large enough.

In a very similar manner one can show that an inequality like (V.73) holds for $\hat{Z}(d+1) > 0$. Hence we have shown that for all $d \geq u$ and $u \in \mathbb{N}$ large enough

$$\mathbf{P}[\hat{M}_{u+1} = \infty \mid \hat{M}_u = d] \geq \kappa > 0.$$

As in the proof of Lemma 12, we may now conclude that $\hat{W}_k$ converges to a random variable $\hat{V}$. We now show that $\hat{V} \in (0,1)$ with probability 0, which implies that $\hat{V} \in \{0,1\}$. As the steps are very similar to (ConvW) in the proof of Lemma 12, we only sketch this proof.

$$\mathbf{P}\big[\hat{V} \in (0,1)\big] = \mathbf{P}\big(\lim_{k\to\infty} \hat{W}_k \in (0,1)\big) \tag{V.77}$$
$$\leq \lim_{k\to\infty} \prod_{m=1}^{\infty} \Big[1 - \mathbf{P}\big[\boldsymbol{X}^{(\cdot)}_{e_{k+m}-1} \neq a \big| \hat{W}_{k+l} \in (0,1), l = 0,\ldots,m-1\big]\Big].$$

Under the condition $\hat{W}_{k+l} \in (0,1), l = 0,\ldots,m-1$, we have in particular that $\hat{W}_{k+l} \leq 1 - \frac{1}{N_b}$, hence, from (V.69) with $\hat{w} := 1 - \frac{1}{N_b}$, we obtain for $m \geq 1$

$$\mathbf{P}\big[\boldsymbol{X}^{(\cdot)}_{e_{k+m}-1} \neq a \big| \hat{W}_{k+l} \in (0,1), l = 0,\ldots,m-1\big] \tag{V.78}$$
$$\leq \Big[\frac{1}{N_b} + \big(1 - \frac{1}{N_b}\big)(1-\varrho)^m + \sum_{j=0}^{\infty} \beta_{k-1+j}\Big]^N.$$

For k large enough, this expression has an upper bound smaller than one. Hence the product in (V.77) vanishes, and $\mathbf{P}(\hat{V} \in (0,1)) = 0$ holds.

Now assume that $\hat{W}_k = \hat{w} \in \{0,1\}$ for $k \geq K$ for some $K \in \mathbb{N}$. Then we see from (V.69) that

$$\big|\boldsymbol{\Pi}_{e_{k+m}}(a;i) - \hat{w}\big| \leq (1-\varrho)^m + \sum_{j=1}^{\infty} \beta_{k+j},$$

which can be made arbitrarily small by choosing m and k large enough. Hence we have $\lim_{k\to\infty} \boldsymbol{\Pi}_{e_k}(a;i) = \hat{w}$, and, as $\hat{w} \in \{0,1\}$ almost surely, the proof is complete. ∎

Proof of Theorem 6: We first show that ϱ_t, defined in (III.3), fulfills the condition of Lemma 13. We have $c_k = e_{k+1} - e_k$, hence for $k \geq 1$

$$1 - \beta_k = \prod_{m=e_k+1}^{e_{k+1}-1} (1-\varrho_m) = \Big((1-x_k)^{\frac{1}{c_k-1}}\Big)^{c_k-1} = 1 - x_k$$

and $\sum_{k=0}^{\infty} \beta_k = \sum_{k=0}^{\infty} x_k < \infty$. From Lemma 13, we now see that under this sequence, the density vector converges. Next we show that ϱ_t also fulfills the sufficient condition for $\mathbf{P}(\tau < \infty) = 1$ of Theorem 1 a).

Observe that

$$\sum_{t=1}^{\infty} \prod_{m=1}^{t} (1-\varrho_m)^L = \sum_{k=1}^{\infty} \sum_{t=e_k}^{e_{k+1}-1} \prod_{m=1}^{t} (1-\varrho_m)^L$$
$$\geq \sum_{k=1}^{\infty} c_k \prod_{m=1}^{e_{k+1}-1} (1-\varrho_m)^L$$
$$\geq \chi^L \cdot \sum_{k=1}^{\infty} c_k (1-\varrho)^{kL} = \infty$$

where we used that $\prod_{l=1}^{k}(1-x_l) \geq \prod_{l=1}^{\infty}(1-x_l) = \chi > 0$ for some $\chi > 0$ as $\sum_{l=1}^{\infty} x_l < \infty$. ∎

VI. Conclusion

In this paper we extend the cross entropy optimization model by a concept of feasibility and desirability of solutions.

We present a precise study of the asymptotic behaviour of the sample process and the sampling density in this generalized model. In particular, we show that depending on the smoothing parameter ϱ_t, different types of convergence may appear. We proved that for a constant smoothing rate, optimal solutions may not be reached with a positive probability, but for a standard test problem the runtime is polynomially bounded with a probability converging to 1.

Our generalized cross entropy model covers standard ant models, therefore our results also complement known results on convergence of ant algorithms.

Future research will look at more general update mechanism with the goal to include the best-so-far update into our model.

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